Understanding Horizontal Geodetic Network Precision and Accuracy Determination Using Least Squares Technique

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Abstract: *Although, there are various software that perform least squares adjustment of horizontal geodetic network so as to determine the precision and accuracy of the network. But the theories and equations that were used to develop the programs which the software are using are not quite understood by various user. This paper presents step by step procedures of horizontal geodetic network precision and accuracy determination using observation equation method of least squares technique. The detailed procedures are enumerated in sequential order for users understanding. The enumerated procedures are also demonstrated using a numeric example.*

Key words: *least squares adjustment, geodetic network, precision, accuracy.*

1. INTRODUCTION:

Least Squares is a tool that helps to analyze and adjust the random errors in survey measurements. It computes adjusted positions using estimated precisions of observations coordinates to reconcile differences between observations and the inverses to their adjusted coordinates. Least Squares also reports the statistics of the adjustments, indicating the strength of a computed position. The strength of the computed position enables the determination of the confident in the computed position and also allows blunder detection. The method of least squares is a rigorous technique that can be applied to the adjustment of horizontal geodetic network to yield the most likely values of the survey measurements.

Horizontal geodetic network is a group of reference stations whose positions as well as coordinates are determined with a very high accuracy. The horizontal geodetic network is classified into three as: first order, second order and third order. The classes as well as the orders are determined by the accuracy with which the nets are established. The precision and the accuracy of the network are normally determined with least squares adjustment technique.

The determination of the precision and accuracy of the horizontal geodetic network using the least squares technique has been very difficult to understand by various researchers. Though, there are software that do this adjustment but the theory as well as the equations that were used to develop the programs and their applications are not quite understood by various users. The difficulty in its application resulted from its matrix nature. The general matrix notation of least squares adjustment is simply the sum of the estimate and the matrix of observations equals to the residual matrix. Unfortunately, to deduce/obtain each of these matrices has been a problem. Previous studies in which least squares adjustment technique was applied never presented the breakdown of how the technique was effected in the studies. To determine the precision and accuracy of the adjusted network, residual matrix has to be obtained and there must be redundant observations as well as degree of freedom.

This paper presents detailed as well as step by step application of observation equation method of least squares adjustment technique for determination of the precision and accuracy of horizontal geodetic networks established using DGPS.

1.1 Observation Equation Method of Least Squares Adjustment

Equations that relate observed quantities to both observational residuals and independent unknown parameters are called observation equations. One equation is written for each observation and for a unique set of unknowns. For a unique solution of unknowns, the number of equations must equal the number of unknowns. Usually, there are more observations (and hence equations) than unknowns, and this permits determination of the most probable values for the unknowns based on the principle of least squares [1].

References [2], [3] explained that, in the observations equation method, the adjusted observations are expressed as a function of the adjusted parameter. The functional relationship between adjusted observations and the adjusted parameters as given in [4] is:

$$L_a = F(X_a) \tag{1}$$

Where, L_a = adjusted observations and X_a = adjusted parameters. Equation (1) is linear function and the general observation equation model was obtained.

To make the matrix expression for performing least squares adjustment, analogy will be made with the systematic procedures. The system of observation equations is presented by matrix notation as [5], [6]:

$$V = AX - L \tag{2}$$

where, A = Design Matrix, X = Vector of Unknowns, L = Calculated Values (l_o) Minus Observed Values (l_b), V = Residual Matrix. That is,

$$V = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_m \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \ X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_m \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} l_1 \\ l_2 \\ \dots \\ l_m \end{pmatrix}$$

The determination of the unknown parameters matrix, X requires the normal matrix, N and the matrix of numeric terms, t to be deduced. For weighted observations, the weight matrix which is always a square matrix also has to be derived.

According to [7], a system of weighted linear observation equations can be expressed in matrix notation as:

$$A^T WAX = A^T WL \tag{3}$$

To make X the subject of the formula, both sides of equation (3) will be divided by $A^T WA$. Thus,

$$X = (A^T WA)^{-1} A^T WL \tag{4}$$

If $A^T WA = N$, normal matrix and $A^T WL = t$, matrix of numeric terms, then equation (4) becomes

$$X = N^{-1} t \tag{5}$$

1.2 Weight of Uncorrelated Observation

The weight, w of an uncorrelated observation as given by [8] is inversely proportional to variance, σ^2. Thus,

$$w = \frac{1}{\sigma^2} \tag{6}$$

The weight matrix of uncorrelated observations is a diagonal matrix such that the off diagonal elements are all zero and it is given as

$$w = \begin{pmatrix} \frac{1}{\sigma_{11}^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma_{12}^2} & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma_{nn}^2} \end{pmatrix} \tag{7}$$

1.3 Precision and Accuracy Determination

The mean as well as the average precision and accuracy of a group of adjusted observations are respectively determined using a posteriori variance and a posteriori standard error. It is to be noted here that variance and standard error are respectively measures of precision and accuracy. The formulae for the computation of a posteriori variance as given by [9] is

$$\hat{\sigma}_o^2 = \frac{V^T WV}{r} \tag{8}$$

while that of the a posteriori standard error as given by [10], [6] is

$$\hat{\sigma}_o = \sqrt{\frac{V^T WV}{r}} \tag{9}$$

Where, $r = n - m$ = degrees of freedom, n = number of observations and m = number of unknown parameters.

The precision and accuracy of the adjusted parameter as well as the adjusted coordinates are obtained from the variance covariance matrix. The precision of the adjusted parameters are the variances as well as the elements of the principal diagonal of the variance covariance matrix. The off-diagonal elements of the variance covariance matrix are the co-variances between particular coordinates. The square root of the variances give the accuracy of the adjusted parameters.

The variance covariance matrix is determined by multiplying the inverse of the normal matrix with the a posteriori variance. Thus,

$$\sum_{XX} = \hat{\sigma}_o^2 N^{-1} \tag{10}$$

Trace is another measure of accuracy. It is the sum of the diagonal elements of the variance covariance matrix, $\sum_{XX}$. It is usually written as $\text{tr}\sum_{XX}$ or sometimes $\text{tr}(\sum_{XX})$, that is,

$$\text{tr}(\sum_{XX}) = \sum \hat{\sigma}_o^2 Q_{nn} \tag{11}$$

According to [11], trace of the variance covariance matrix can be interpreted as a measure of the overall accuracy of the associated vector of random variates.

The models for the computation of the variance, σ_{xi}^2 and the standard error, σ_{xi} of the adjusted coordinates are given by [6] as

$$\sigma_{xi}^2 = \hat{\sigma}_o^2 Q_{nn} \tag{12}$$

$$\sigma_{xi} = \sqrt{\hat{\sigma}_o^2 Q_{nn}} \tag{13}$$

where, Q_{nn} = principal diagonal element of the inverse of the normal matrix.

Error ellipses are usually computed during adjustment of horizontal or three - dimensional network. They allow a convenient way for interpretation of the directional station position accuracy. Error ellipses are also used in optimising a network. [12] gave the semi-major axis $\sigma_{x^1}^2$, semi-minor axis $\sigma_{y^1}^2$ and the orientation of the error ellipse θ as

$$\sigma_{x^1}^2 = \frac{\sigma_x^2 + \sigma_y^2}{2} + \left[\frac{(\sigma_x^2 - \sigma_y^2)^2}{4}\right] + \sigma_{xy}^2$$
$$\sigma_{y^1}^2 = \frac{\sigma_x^2 + \sigma_y^2}{2} - \left[\frac{(\sigma_x^2 - \sigma_y^2)^2}{4}\right] + \sigma_{xy}^2 \tag{14}$$

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2} \tag{15}$$

2. STEPS TO BE CONSIDERED WHEN DETERMINING THE PRECISION AND ACCURACY OF HORIZONTAL GEODETIC NETWORK USING OBSERVATION EQUATION METHOD OF LEAST SQUARES ADJUSTMENT TECHNIQUE

The following are the steps to be followed or considered when determining the precision and accuracy of a horizontal geodetic network established with DGPS:

1. There must be not less than two fixed controls. That is, some of the network points have to be observed from not less than two controls and the data acquired with respect to any of the controls/base station should be processed separately to obtain the coordinates of the new points.
2. Having obtained the coordinates of the new points, deduce the normal as well as the observation equations. The number of observation equations must be equal to the number of baseline.
3. From the deduced observation equations, derive the coefficient as well as the design matrix, *A,* observation matrix, *L*, residual matrix, *v* and matrix of unknown parameters, *X*.
4. Also, deduce the weight matrix using (6). For DGPS observation, the variances of the baseline vectors are obtained from the variance covariance matrix of the processed DGPS data.
5. Having deduced the above stated matrices, compute the unknown parameters using (5).
6. To determined the estimate as well as the most probable values, the design matrix is used to multiply the computed values of the unknown parameters.
7. Having determined the estimates, the next step is to evaluate the residuals. The computation of the residual is done by finding the differences between the estimates and the observations using (2).
8. Since the residuals have been computed, the precision of the adjusted observations is determined using (8). To compute the precision of the adjusted observations, the degree of freedom must be evaluated.
9. Having determined the precision of the adjusted observations, their mean accuracy is determined using (9). Thus, the square root of the determined precision.
10. The accuracy of the adjusted network can also be determined using (11).
11. Having determined the precision and accuracy of the adjusted observations, the next step is to determined the precision and accuracy of the adjusted coordinates using (12) and (13) respectively.
12. To compute the semi-major axis, semi-minor axis and the orientation of the error ellipse, (14) and (15) are respectively used.

2.1 Numerical Application in Horizontal Geodetic Network

Fig.1 shows two network stations, A and B observed with respect to two controls, S and T using DGPS. The rectangular coordinates of the stations as determined with respect to the control stations and their corresponding variances as obtained from the variance covariance matrices of the processed DGPS data are as shown in table 1. The rectangular coordinates of control stations S and T are respectively 251374.548mN, 350472.960mE and 251441.978mN, 354095.611mE. Using least squares technique, determine the most probable coordinates for stations A and B, the precision and accuracy of the observations as well as those of the adjusted coordinates. Also, compute the error ellipses of the stations positions.

Table 1: Observed Coordinates and their Corresponding Variances

Base Station	Rover Station	Coordinates (m)		Variance (m)	Base Station	Rover Station	Coordinates (m)		Variance (m)
S	A	Northing	250852.942	0.0000577	T	A	Northing	250852.949	0.0000465
		Easting	352598.178	0.0000314			Easting	352598.188	0.0000554
	B	Northing	252127.392	0.0000247		B	Northing	252127.398	0.0000338
		Easting	352572.216	0.0000822			Easting	352572.226	0.0000709

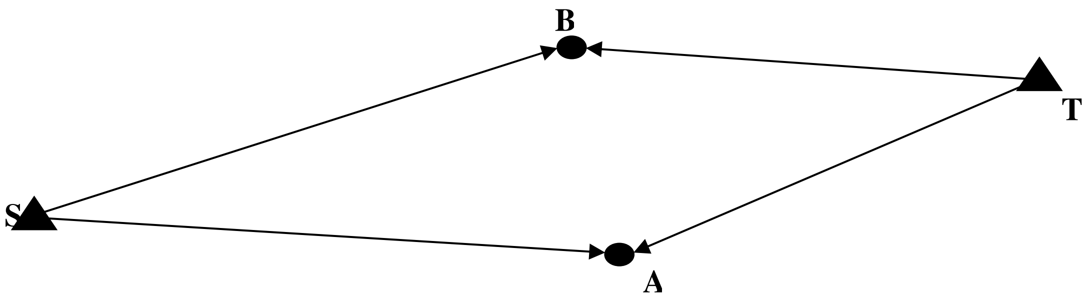

Fig. 1: Observed Network Points

Solution

Table 2: Determination of baseline vector

STATION FROM	ΔN (m)	ΔE (m)	COORDINATES NORTHING (m)	COORDINATES EASTING (m)	STATION TO
			251374.548	350472.960	S
S	-521.606	2125.218	250852.942	352598.178	A
S	752.844	2099.256	252127.392	352572.216	B
			251441.978	354095.611	T
T	-589.029	-1497.423	250852.949	352598.188	A
T	685.420	-1523.385	252127.398	352572.226	B

Deduction of observation equations

$$A_N = S_N + \Delta N_{SA} + V_1$$
$$A_E = S_E + \Delta E_{SA} + V_2$$
$$B_N = S_N + \Delta N_{SB} + V_3$$
$$B_E = S_E + \Delta E_{SB} + V_4$$
$$A_N = T_N + \Delta N_{TA} + V_5$$
$$A_E = T_E + \Delta E_{TA} + V_6$$
$$B_N = T_N + \Delta N_{TB} + V_7$$
$$B_E = T_E + \Delta E_{TB} + V_8$$

Deduction of design matrix, *A*, observation matrix, *L*, matrix of unknown parameters, *X*, weight matrix, *w* and residual matrix, *v*

$$A=\begin{pmatrix} A_N & 0 & 0 & 0 \\ 0 & A_E & 0 & 0 \\ 0 & 0 & B_N & 0 \\ 0 & 0 & 0 & B_E \\ A_N & 0 & 0 & 0 \\ 0 & A_E & 0 & 0 \\ 0 & 0 & B_N & 0 \\ 0 & 0 & 0 & B_E \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L=\begin{pmatrix} S_N+\Delta N_{SA} \\ S_E+\Delta E_{SA} \\ S_N+\Delta N_{SB} \\ S_E+\Delta E_{SB} \\ T_N+\Delta N_{TA} \\ T_E+\Delta E_{TA} \\ T_N+\Delta N_{TB} \\ T_E+\Delta E_{TB} \end{pmatrix}=\begin{pmatrix} 250852.942 \\ 352598.178 \\ 252127.392 \\ 352572.216 \\ 250852.949 \\ 352598.188 \\ 252127.398 \\ 352572.226 \end{pmatrix}, \quad V=\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{pmatrix},$$

$$W=\begin{pmatrix} 17331.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 31847.13 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 40485.83 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12165.45 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 21505.38 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 18050.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 29585.80 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14104.37 \end{pmatrix}, \quad X=\begin{pmatrix} A_N \\ A_E \\ B_N \\ B_E \end{pmatrix}$$

Computation of the unknown parameters, X using (5)

$$\text{Normal matrix, } N=A^TWA=\begin{pmatrix} 38836.40 & 0 & 0 & 0 \\ 0 & 49897.68 & 0 & 0 \\ 0 & 0 & 70071.63 & 0 \\ 0 & 0 & 0 & 26269.82 \end{pmatrix},$$

$$N^{-1}=(A^TWA)^{-1}=\begin{pmatrix} 0.0000257 & 0 & 0 & 0 \\ 0 & 0.0000200 & 0 & 0 \\ 0 & 0 & 0.0000143 & 0 \\ 0 & 0 & 0 & 0.0000381 \end{pmatrix}$$

$$\text{Matrix of numeric terms}, t=A^TWL=\begin{pmatrix} 9742225063.763 \\ 17593829570.669 \\ 17666977195.008 \\ 9262009666.744 \end{pmatrix}$$

$$\therefore X=N^{-1}t=\begin{pmatrix} 0.0000257 & 0 & 0 & 0 \\ 0 & 0.0000200 & 0 & 0 \\ 0 & 0 & 0.0000143 & 0 \\ 0 & 0 & 0 & 0.0000381 \end{pmatrix}\times\begin{pmatrix} 9742225063.763 \\ 17593829570.669 \\ 17666977195.008 \\ 9262009666.744 \end{pmatrix}=\begin{pmatrix} 250852.946 \\ 352598.182 \\ 252127.395 \\ 352572.221 \end{pmatrix}$$

Computation of the most probable coordinates for stations A and B

$$AX=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\times\begin{pmatrix} 250852.946 \\ 352598.182 \\ 252127.395 \\ 352572.221 \end{pmatrix}=\begin{pmatrix} 250852.946 \\ 352598.182 \\ 252127.395 \\ 352572.221 \\ 250852.946 \\ 352598.182 \\ 252127.395 \\ 352572.221 \end{pmatrix}$$

Therefore, the most probable coordinates for stations A and B are respectively 250852.946mN, 352598.182mE and 252127.395mN, 352572.221mE.

Evaluation of residuals, v using (2)

$$V = AX - L = \begin{pmatrix} 250852946 \\ 352598182 \\ 252127395 \\ 352572221 \\ 250852946 \\ 352598182 \\ 252127395 \\ 352572221 \end{pmatrix} - \begin{pmatrix} 250852.942 \\ 352598.178 \\ 252127.392 \\ 352572.216 \\ 250852.949 \\ 352598.188 \\ 252127.398 \\ 352572.226 \end{pmatrix} = \begin{pmatrix} 0.004 \\ 0.004 \\ 0.003 \\ 0.005 \\ -0.003 \\ -0.006 \\ -0.003 \\ -0.0005 \end{pmatrix}$$

Computation of the precision and accuracy of the adjusted network using (8) and (9) respectively.
Since there are 4 unknown and 8 observations, the degree of freedom, r is 4.

$$\hat{\sigma}_o^2 = \frac{2.891}{4} = 0.723\text{m}^2$$

$$\hat{\sigma}_o = \sqrt{\frac{2.891}{4}} = \sqrt{0.723} = 0.850\text{m}$$

Therefore, the precision and accuracy of the adjusted network are respectively 0.723m^2 and 0.850m.

Computation of the accuracy of the adjusted network using (11). Thus, the trace of the variance covariance matrix.

$$\sum_{XX} = 0.723\begin{pmatrix} 0.0000257 & 0 & 0 & 0 \\ 0 & 0.0000200 & 0 & 0 \\ 0 & 0 & 0.0000143 & 0 \\ 0 & 0 & 0 & 0.0000381 \end{pmatrix}$$

$$= \begin{pmatrix} 0.0000186 & 0 & 0 & 0 \\ 0 & 0.0000145 & 0 & 0 \\ 0 & 0 & 0.0000103 & 0 \\ 0 & 0 & 0 & 0.0000275 \end{pmatrix}$$

Therefore, the accuracy of the adjusted network using equation (10) is

$$\sum_{XX} = 0.0000186 + 0.0000145 + 0.0000103 + 0.0000275 = 0.0000709\text{m}^2$$

$$= \sqrt{0.0000709} = 0.00842\text{m}$$

Computation of the precision and accuracy of the adjusted coordinates using (12) and (13).

$$\text{Station A precision,} \begin{pmatrix} A_N \\ A_E \end{pmatrix} = 0.723\begin{pmatrix} 0.0000257 \\ 0.0000200 \end{pmatrix} = \begin{pmatrix} 0.0000186 \\ 0.0000145 \end{pmatrix}$$

$$\text{Station B precision,} \begin{pmatrix} B_N \\ B_E \end{pmatrix} = 0.723\begin{pmatrix} 0.0000143 \\ 0.0000381 \end{pmatrix} = \begin{pmatrix} 0.0000103 \\ 0.0000275 \end{pmatrix}$$

$$\text{Station A accuracy,} \begin{pmatrix} A_N \\ A_E \end{pmatrix} = \begin{pmatrix} \sqrt{0.0000186} \\ \sqrt{0.0000145} \end{pmatrix} = \begin{pmatrix} 0.00431 \\ 0.00381 \end{pmatrix}$$

$$\text{Station } B \text{ accuracy,} \begin{pmatrix} B_N \\ B_E \end{pmatrix} = \begin{pmatrix} \sqrt{0.0000103} \\ \sqrt{0.0000275} \end{pmatrix} = \begin{pmatrix} 0.00321 \\ 0.00524 \end{pmatrix}$$

Therefore, the precision of northing and easting coordinates of stations A and B are respectively $0.0000186, 0.0000145$ and $0.0000103, 0.0000275$ while their respective accuracy are $0.00431, 0.00381$ and $0.00321, 0.00524$.

Computation of error ellipses of the stations positions using (14) and (15) respectively.

For station A

$$\sigma_{x^1}^2 = \frac{0.0000186 + 0.0000145}{2} + \left[\frac{(0.0000186 - 0.0000145)^2}{4}\right] + 0$$

$$= 0.000033\text{m}^2$$

$$\sigma_{y^1}^2 = \frac{0.0000186 + 0.0000145}{2} - \left[\frac{(0.0000186 - 0.0000145)^2}{4}\right] + 0$$

$$= 0.000033\text{m}^2$$

$$\sigma_{x^1} = \sqrt{0.000033} = 0.00575\text{m}$$

$$\sigma_{y^1} = \sqrt{0.000038} = 0.00575\text{m}$$

$$\tan 2\theta = \frac{0}{0.0000186 - 0.0000145} \Rightarrow \theta = 0^o$$

The same procedure was repeated for station *B* and the following were obtained

$$\sigma_{x^1} = 0.00615\text{m}$$

$$\sigma_{y^1} = 0.00615\text{m}$$

$$\theta = 0^o$$

Therefore, the semi-major axis, semi-minor axis and the orientation of positions *A* and *B* are respectively $0.00575\text{m}, 0.00575\text{m}, 0^o$ and $0.00615\text{m}, 0.00615\text{m}, 0^o$.

It is to be noted here that since the observations are uncorrelated, there is no covariance between stations *A* and *B*. Thus, the orientations of the error ellipses are zero.

3. CONCLUSION:

The determination of horizontal geodetic (DGPS) network precision and accuracy using the least squares method which is rigorous enables the reliability as well as the order or class of the network to be determined. The advent of computer system which gave rise to the development of software has simplified these rigorous computations. But as the theories as well as the equations which were used to develop the programs which the software are using are not quite understood by various users, this paper has given the step by step application of this rigorous method as well as its numeric application in horizontal geodetic network precision and accuracy determination. Considering the enumerated procedures as well as the given numeric example, the theories and the equations as well as the processing procedures of the least squares adjustment software have been explained in detail for users understanding.

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