Analysis of error propagation and sensitivity of logistic map

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Abstract: *The logistic maps are commonly used in, for example, chaos based cryptography. However, its properties do not render a safe construction of encryption algorithms. The purpose of this paper is to measure the Lyapunov exponent $\lambda(x_0)$, this is different at different value for β and r, which indicates the instability of this fixed point for most other points x between 0 and 1, on computed Lyapunov exponent $\lambda(x_0)$ is positive and negative at different initial points. The greatest problem that computers are confront with when dealing with chaos is the extreme sensitivity of an iterator. The phenomenon of sensitivity, however, magnifies even the smallest error and sensitive depended on initial conditions, i.e. instability is a more sensitive and unstable behavior. In superior orbit, we see that the range of instability of logistic map causing on over flow error, chaotic behavior of logistic map disappears in certain cases.*

Key Words: *Logistic maps; Lyapunov exponent; chaotic behavior; mann iterations; stability; errors.*

1. INTRODUCTION AND PRELIMINARIES:

The maximal Lyapunov exponent defined as follows:

$$\lambda = \lim_{t\to\infty}\lim_{\delta Z_0\to 0}\frac{1}{t}\ln\frac{|\delta Z(t)|}{|\delta Z_0|}$$

The limit $\delta Z_0 \to 0$ ensures the validity of the linear approximation at any time.

For discrete time system (maps or fixed point iterations) $x_{n+1} = f(x_n)$ for n = 0, 1, 2, . . ., for an orbit starting with x_0 this translates as:

$$\lambda(x_0) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln|f'(x_{k-1})| \quad (1.1)$$

Alexander Michailowitsh Lyapunov, introduced the concept of Lyapunov exponent $\lambda(x_0)$ for study to instability of fixed point and unstable chaotic behavior of logistic map (see [19]).

Now again we consider an equation $\ln c = \frac{1}{n}\ln|\frac{E_n}{E_0}|$, first we rewrite the relative growth of the error (the total error amplification factor $|\frac{E_n}{E_0}|$) (see [11]) after n steps as a product

$$|\frac{E_n}{E_0}| = |\frac{E_n}{E_{n-1}}|.|\frac{E_{n-1}}{E_{n-2}}|. . . |\frac{E_1}{E_0}|$$

In formula,

$$\frac{1}{n}\ln|\frac{E_n}{E_0}| = \frac{1}{n}\ln|\frac{E_n}{E_{n-1}}.\frac{E_{n-1}}{E_{n-2}}. . . \frac{E_1}{E_0}|$$

$$= \frac{1}{n}\sum_{k=1}^{n}\ln|\frac{E_k}{E_{k-1}}|. \quad (1.2)$$

The reasoning above directly leads to the concept of Lyapunov exponent $\lambda(x_0)$. Summing up the logarithms of the amplifications factors surely avoids the overflows problems. By definition of the error terms

$$\frac{E_k}{E_{k-1}} = \frac{f(x_{k-1}+E_{k-1}) - f(x_{k-1})}{E_{k-1}}$$

Moreover, from calculus obtain

$$\lim_{E_0\to 0} \frac{E_k}{E_{k-1}} = f'(x_{k-1}).$$

Thus,

$$\lim_{E_0\to 0} \frac{1}{n}\sum_{k=1}^{n} \ln \left| \frac{E_k}{E_{k-1}} \right| = \frac{1}{n}\sum_{k=1}^{n} \ln |f'(x_{k-1})|.$$

Now letting $n\to \infty$ we obtain the Lyapunov exponent $\lambda(x_0)$ is

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \ln |f'(x_{k-1})| \,. \qquad (1.3)$$

The concept of Lyapunov exponent $\lambda(x_0)$ quantifies the average growth of infinitesimally small errors in the initial point x_0. The concept of Lyapunov exponent $\lambda(x_0)$ is a powerful experimental device to separate unstable chaotic behavior from that which is stable and predictable and to measure these properties. Especially where $\lambda(x_0) > 0$ is large sensitivity with respect to small changes in initial conditions is large. Therefore, that it applies to many interesting dynamical systems, in mathematical and the sciences. It has become one of these to measuring; evaluating and detecting chaotic behavior (see [11]).

Mathematically, logistic map in general is given by the equation $x_{n+1} = rx_n(1 - x_n)$, where x_n lies between zero and one, and represents the point at iteration n. Therefore, x_0 represents the initial point (at iteration 0) and r represents a positive parameter set, in other way logistic map have represented one may refer to [5] and [15]. In recent years, chaotic dynamics has used in cryptography using logistic map [9, 10]. In addition, for optimization of global searching capacity, chaos optimization has used. For details, one may refer to [17] and several references thereof. For various other applications and complexity of logistic map, one may refer to Ausloos [1], Bunde and Havlin [2], Crownover [3], Devaney [4], Holmgren [5], Kumar [6], Kumar and Rani [7], Peitgen, Jurgens and Saupe [11], Peitgen and Jurgens [12], Hassan and Mohammad [16], Rani and Agrawal [15] and [18]. Our main aim of this paper to computes the Lyapunov exponent $\lambda(x_0)$ expand for study to unstable chaotic behavior and sensitivity of logistic map via mann iteration for various values of β and r for different four initial point used by mat lab software which is leads to conclusion of [15].

Definition 1.1. The sequence { x_n } constructed above will be called Mann sequence of iterates or superior sequence of iterates. We denote it by $SO(f, x_0, \beta_n)$.

Notice that $SO(f, x_0, \beta_n)$ with $\beta_n = 1$ is $O(f, x_0)$, i.e, the Mann iteration at $\beta = 1$ is the function iteration. This procedure is essentially due to W.R. Mann [8]. We shall consider the Mann or superior successive approximation method generally for $\beta_n = \beta$ (see [11, 13, 14, 15]).

Superior stable set: In all that follows, R^1stands for the set of real numbers endowed with the usual metric and S^1 for a subspace of R^1 (see [11, 13, 14, 15]).

Definition 1.2. Let $S^1 \subset R^1$ and let f: $S^1 \to S^1$ and p be a periodic point of f with prime period k for a point $x_0 \in S^1$ and p ϵ [0, 1], construct a sequence $\{x_n: n = 1, 2 \ldots\}$ such that

$x_k = (1-\beta)x_0 + \beta f^k(x_0)$,
$x_{2k} = (1-\beta)x_k + \beta f^k(x_k), \ldots,$
$x_{nk} = (1-\beta)x_{(n-1)k} + \beta f^k(x_{(n-1)k}), \ldots,$

Then x_0 will be called Mann forward asymptotic or superior forward asymptotic to p and the sequence { x_{nk} } converges to p. The Mann stable set or superior stable set to p, denoted by $W^{ss}(p)$, consists of all points, which are superior forward asymptotic to p [11, 13,14,15].

2. ANALYSIS OF ERROR PROPAGATION OF LOGISTIC MAP:

The equation (1.2) will guide us to appropriate definition

$$E_{k+1} = g(f(x_k+ \varepsilon), x_k+ \varepsilon) - g(f(x_k), x_k)$$
$$= \beta r(x_k+ \varepsilon)(1-(x_k+ \varepsilon)) + (1-\beta).(x_k+ \varepsilon) - (\beta r\, x_k (1- x_k) + (1-\beta)\, x_k)$$

take $E_0 = \varepsilon$ and $E_{k+1} \cong E_k$, after simplification we get

$$\frac{E_k}{E_0} = [(1-2x_k)\, \beta r + (1-\beta) - E_0\, \beta r]$$

$$\frac{1}{n} \ln \left|\frac{E_k}{E_0}\right| = \frac{1}{n} \ln |[(1-2\, x_k)\, \beta r + (1-\beta) - E_0\, \beta r]|. \quad (2.1)$$

In summary, we have arrived at an improved and feasible method to compute the Lyapunov exponent more reliably by averaging over quadratic iterator $x \rightarrow (\beta r\, x\, (1- x) + (1-\beta)\, x)$.

$$\frac{1}{n} \ln \left|\frac{E_k}{E_0}\right| \cong \frac{1}{n}\sum_{k=1}^{n} \ln \left| \frac{E_k}{E_0}\right|.$$

The results for the initial values used in table 1 and Table 2. An error of 0.001 has used in each iteration. As the number of iterations grows the exponent converges to different fixed points by equation (2.1) for different values of β and r, i.e., for β = 0.9, 0.64, 0.17, 0.1, for x_0 = 0.15 is 1.1788, 0.5851, -0.7328, -1.2501 for 7 iterations respectively and other results for initial values x_0 = 0.25, 0.35, 0.5 (see in table 1 and table 2).

Measuring the exponents in this more careful analysis results in the numbers of iterations with the help of equation of quadratic iterator $g(f(x_k), x_k) = (\beta r\, x_k (1- x_k) + (1-\beta)\, x_k)$

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \ln |g'((f x_k), x_k)|$$
$$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \ln |\beta r(1-2x_k) + (1-\beta)| \quad (2.2)$$

With this we have succeeded in quantifying the sensitive dependence on initial conditions for the quadratic iterator $x \rightarrow (\beta r\, x\, (1- x) + (1-\beta)\, x)$. Now we are compare the sensitivity found here with that in other chaotic system (see all figure 1, 2, 3 … 24). The experiment in figure clearly shows that such a simple law cannot be expected for the error propagation of the quadratic iterator $x \rightarrow (\beta r\, x\, (1- x) + (1-\beta)\, x)$.

On the other hand, during the first 100 iteration or so, the error must have grown more or less uniformly. This motivates another experiment. We count the number of iterations necessary for the error to exceed a certain upper limit for various initial values and for different initial errors. We choose four initial values, 0.15, 0.25, 0.35 and 0.5, for fix value of β and r and four possible initial errors to arrive at 32 cases. We shows the Table 1 and Table 2 documents the results

Table 1:

X_0	Initial Error E_0	No. of ite.	β = 0.9, r = 3.8705		β = 0.64, r = 5.0015	
			λ(x₀)	$\frac{1}{n}\sum_{k=0}^{n} \ln \left\| E_k/E_0 \right\|$	**λ(x₀)**	$\frac{1}{n}\sum_{k=0}^{n} \ln \left\| E_k/E_0 \right\|$
0.15	0.001	7	0.0092	1.1788	0.32789	0.5851
0.15	0.0001	13	0.1579	0.3939	0.2764	0.6410
0.15	0.00001	19	0.0369	1.1856	0.2634	0.4337
0.15	0.000001	25	0.1076	0.4182	0.2092	1.0312
0.25	0.001	31	0.1179	-0.5653	-0.0049	-0.3705
0.25	0.0001	37	0.1317	0.9351	-0.0404	1.0798
0.25	0.00001	43	0.1042	-0.5580	0.0028	0.0464
0.25	0.000001	49	0.1158	0.9483	-0.0429	-2.7117
0.35	0.001	55	0.0378	-2.8388	-0.0885	-2.1996
0.35	0.0001	61	0.0685	-2.5426	-0.1058	-2.7786
0.35	0.00001	67	0.0404	-2.8365	-0.0698	-2.3773
0.35	0.000001	73	0.0658	-2.5517	-0.0821	-2.7117
0.5	0.001	76	0.1013	0.4055	-0.0484	0.0426
0.5	0.0001	82	0.0261	1.1871	-0.0595	0.8968
0.5	0.00001	88	0.0469	0.4320	-0.0606	-0.3679
0.5	0.000001	97	0.0379	-2.6853	-0.0698	-2.7116

Table 2:

X_0	Initial Error E_0	No. of ite.	β = 0.1, r = 18.9000		β = 0.17,		
			$\lambda(x_0)$	$\frac{1}{n}\sum_{k=0}^{n} \ln \lvert {E_k}/{E_0} \rvert$	$\lambda(x_0)$	$\frac{1}{n}\sum_{k=0}^{n} \ln \lvert {E_k}/{E_0} \rvert$	r =
0.15	0.001	7	-0.2384	-0.7328	-0.0283	-1.2501	12.9599
0.15	0.0001	13	-0.2431	-0.8183	0.0514	-1.2937	
0.15	0.00001	19	-0.2401	-0.7969	0.0251	-1.3125	
0.15	0.000001	25	-0.2389	-0.7917	0.0092	-1.3277	
0.25	0.001	31	-0.2160	-0.7917	-0.0997	-0.8954	12.3053
0.25	0.0001	37	-0.2192	-0.7901	-0.0969	-0.9046	
0.25	0.00001	43	-0.2215	-0.7900	-0.0949	-0.9112	
0.25	0.000001	49	-0.2230	-0.7900	-0.0933	-0.9153	
0.35	0.001	55	-0.2887	-0.7918	-0.1968	-0.8432	11.8311
0.35	0.0001	61	-0.2835	-0.7902	-0.1944	-0.8414	
0.35	0.00001	67	-0.2792	-0.7900	-0.1925	-0.8413	
0.35	0.000001	73	-0.2756	-0.7900	-0.1909	-0.8413	
0.5	0.001	76	-0.2352	-0.7919	-0.0462	-0.6279	12.9692
0.5	0.0001	82	-0.2352	-0.7902	-0.0482	-0.6264	
0.5	0.00001	88	-0.2353	-0.7900	-0.0500	-0.6263	
0.5	0.000001	97	-0.2353	-0.7900	-0.0516	-0.6260	

Table 3:

$\lambda(x_0) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\ln\lvert\beta r(1-2x_k)+(1-\beta)\rvert$, β = 0.9, r = 3.1789					$\lambda(x_0) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\ln\lvert\beta r(1-2x_k)+(1-\beta)\rvert$, β = 0.9, r = 3.8705			
iteration	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5
10	0.0089	-0.2175	-0.0398	-0.3451	0.0396	0.1442	0.1834	0.0629
100	-0.0356	-0.0622	-0.0398	-0.0775	0.0524	0.0305	0.0703	0.0533
1000	-0.0393	-0.0420	-0.0398	-0.0435	0.0344	0.0221	0.0306	0.0176
8000	-0.0397	-0.0401	-0.0398	-0.0402	0.0299	0.0271	0.0234	0.0239
9000	-0.0397	-0.0400	-0.0398	-0.0402	0.0297	0.0272	0.0249	0.0247
10000	-0.0397	-0.0399	-0.0398	-0.0401	0.0287	0.0277	0.0257	0.249

Table 4:

$\lambda(x_0) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\ln\lvert\beta r(1-2x_k)+(1-\beta)\rvert$, β = 0.9, r = 3.8706					$\lambda(x_0) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\ln\lvert\beta r(1-2x_k)+(1-\beta)\rvert$, β = 0.9, r = 3.8800			
iteration	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5
10	0.0411	0.1440	0.1826	0.0636	0.1326	0.1239	-0.0019	0.1159
100	0.0613	0.0401	0.0443	0.1359	0.1723	0.1650	0.1550	0.1619
1000	0.0418	0.0327	0.0289	0.0432	0.1557	0.1596	0.1567	0.1665
10000	0.0318	0.0323	0.0338	0.0329	0.1616	0.1616	0.1619	0.1610

We notice that for $3.8705 \le r \le 3.8800$, $\forall$ x ϵ [0, 1], β = 0.9, exponent $\lambda(x_0)$ is positive, its shows that chaotic behavior of logistic map via mann iteration is unstable and predictable and to measure these properties. Especially where $\lambda(x_0) > 0$ is large, sensitivity with respect to small change in initial condition is large.

We verified that chaotic behavior loses its stable behavior for $r \le 3.1789$, its shows unstable till 10 iterations for $x_0 = 0.15$ after that shows stable behavior and it's also shows non sensitive for all x belonging to [0, 1], see Table 3 and Table 4, and see the behavior of the map in Figure 13, 14, 15, 16.

Table 5:

$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \ln\lvert\beta r(1-2x_k) + (1-\beta)\rvert$, **β** = 0.64, r = 4.3160					$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \ln\lvert\beta r(1-2x_k) + (1-\beta)\rvert$, β = 0.64, r = 5.0015			
iteration	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5
10	0.0704	-0.3835	0.1001	-0.3070	0.2239	0.1028	0.1149	-0.0912
100	-0.2948	-0.3539	-0.2549	-0.3459	0.0257	-0.0609	-0.0559	-0.0502
1000	-0.3449	-0.3508	-0.3409	-0.3500	-0.0490	-0.0554	-0.0570	-0.0565
8000	-0.3498	-0.3505	-0.3493	-0.3504	-0.0552	-0.0560	-0.0562	-0.0562
9000	-0.3499	-0.3505	-0.3494	-0.3504	-0.0552	-0.0560	-0.0562	-0.0561
10000	-0.3499	-0.3505	-0.3495	-0.3504	-0.0554	-0.0560	-0.0562	-0.0561

In Table 5, for 0< r ≤ 4.3160, Lyapunov exponent $\lambda(x_0)$ is positive till 10 iterations for x_0 = 0.15 after that shows stable behavior and it also shows not more sensitive and convergent to fixed-point, for 0 < r ≤ 5.0015, it shows for all initial points exponent is negative its means stable and less sensitive for 4.3160 ≤ r ≤ 5.0015, it shows cyclic behavior for all x belonging to [0, 1]. Cyclic behavior of the function at β = 0.64 has been shown in Figure 1, 2, 3, 4 respectively.

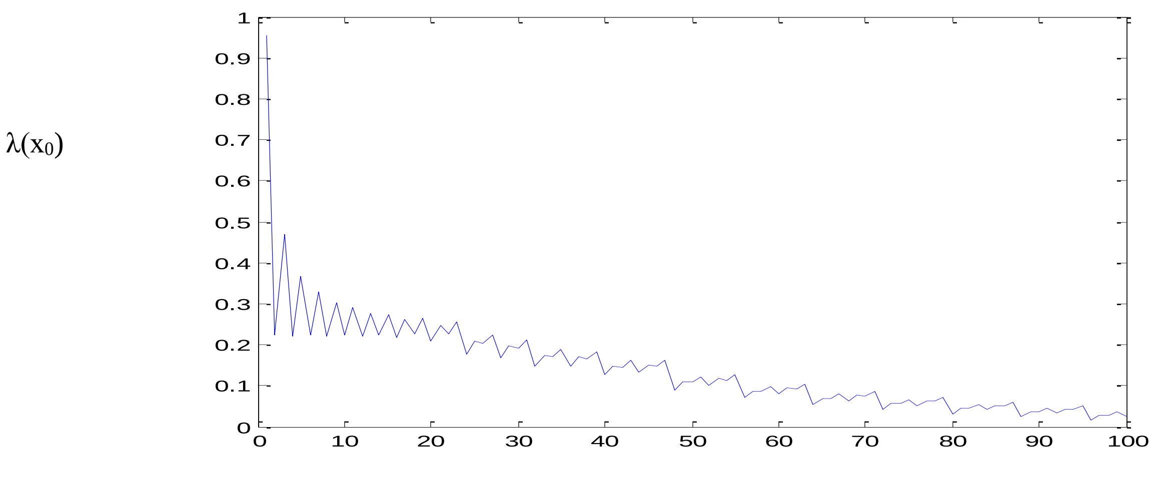


Figure 1. (β = 0.64, r = 5.0015, x_0= 0.15)

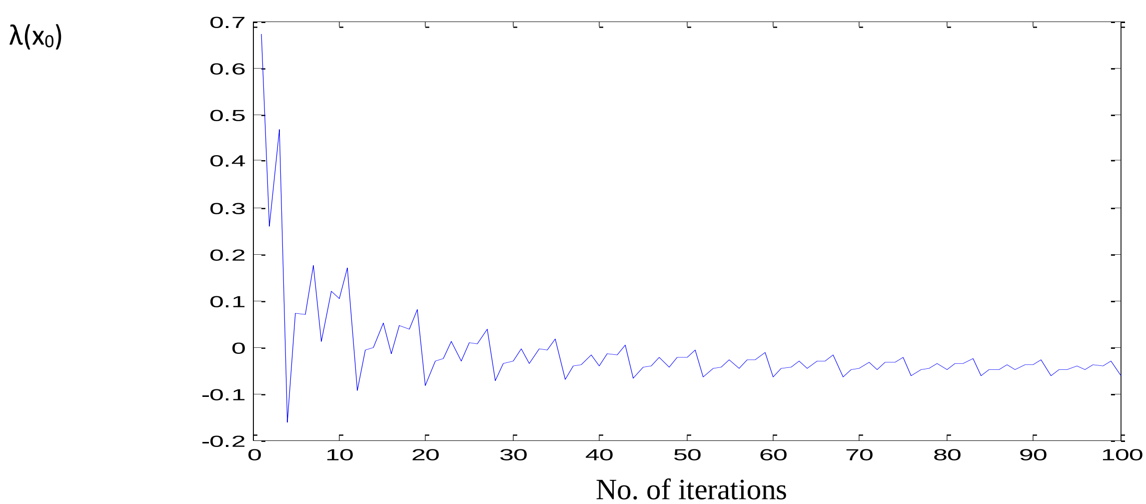


Figure 2. (β = 0.64, r = 5.0015, x_0= 0.25)

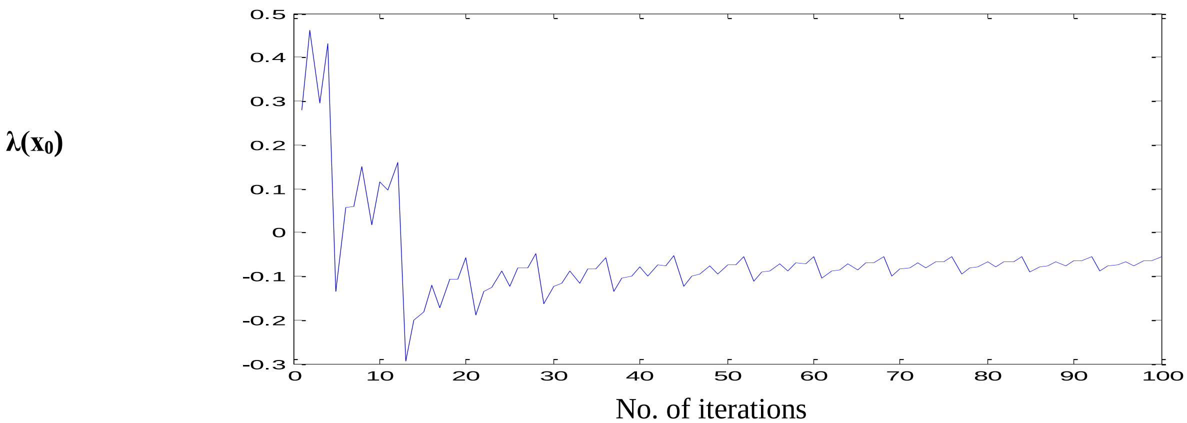


Figure 3. (β = 0.64, r = 5.0015, x_0= 0.35)

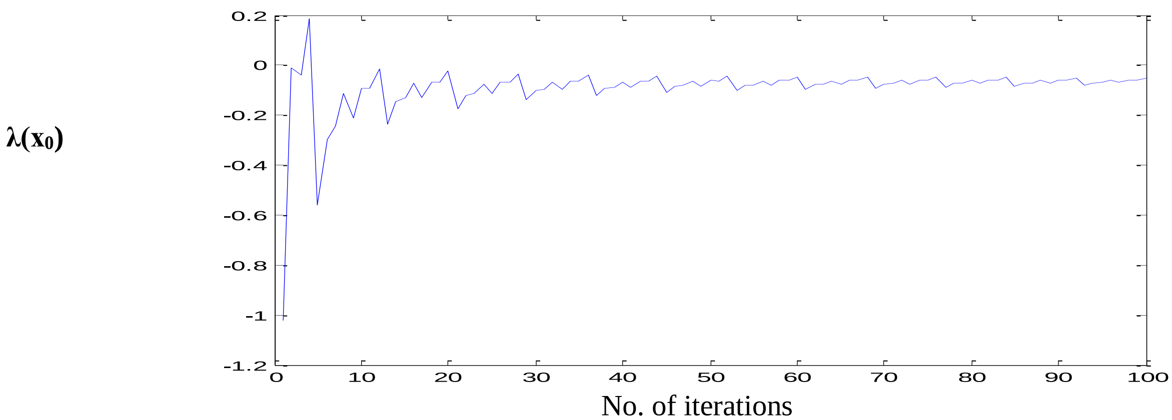


Figure 4. (β = 0.64, r = 5.0015, x_0= 0.5)

Table 6:

$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \ln\lvert\beta r(1-2x_k)+(1-\beta)\rvert$, β = 0.17, r = 12.9599					$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \ln\lvert\beta r(1-2x_k)+(1-\beta)\rvert$, **β** = 0.17, r = 12.9692			
iteration	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5
10	0.1007	-0.1973	-0.1461	0.0014	0.1024	-0.1993	-0.1458	0.0002
100	-0.0435	-0.0904	-0.0837	-0.0500	-0.0459	-0.0936	-0.0866	-0.0530
1000	-0.0690	-0.0737	-0.0730	-0.0697	-0.0726	-0.0774	-0.0767	-0.0733
8000	-0.0715	-0.0721	-0.0719	-0.0716	-0.0752	-0.0758	-0.0757	-0.0753
9000	-0.0715	-0.0720	-0.0719	-0.0716	-0.0752	-0.0758	-0.0757	-0.0753
10000	-0.0716	-0.0720	-0.0719	-0.0717	-0.0752	-0.0757	-0.0757	-0.0753

Table 7:

$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \ln\lvert\beta r(1-2x_k)+(1-\beta)\rvert$, **β** = 0.17, r = 11.8311					$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \ln\lvert\beta r(1-2x_k)+(1-\beta)\rvert$, **β** = 0.17, r = 12.3053			
iteration	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5
10	-0.1343	-0.1343	-0.2107	-0.0880	-0.1286	-0.1574	-0.3056	-0.1735
100	-0.0872	-0.0872	-0.0951	-0.0823	-0.1693	-0.1710	-0.1860	-0.1729
1000	-0.0819	-0.0819	-0.0827	-0.0814	-0.1725	-0.1726	-0.1741	-0.1728
8000	-0.0814	-0.0814	-0.0815	-0.0813	-0.1728	-0.1728	-0.1729	-0.1728
9000	-0.0814	-0.0814	-0.0815	-0.0813	-0.1728	-0.1728	-0.1729	-0.1728
10000	-0.0814	-0.0814	-0.0815	-0.0813	-0.1728	-0.1728	-0.1729	-0.1728

In Table 6, for r = 12.9599, r = 12.9692, shows exponent is positive till 10 iterations at $x_0 = 0.15$ and $x_0 = 0.5$, it show instability after that for all x belonging to [0, 1], Lyapunov exponent $\lambda(x_0)$ of logistic map via mann iteration is negative, it show not more sensitive and convergent to fixed point at initial points. In Table 7, for $0 < r \leq 11.8311$, for all x belonging to [0, 1], Lyapunov exponent $\lambda(x_0)$ of logistic map via mann iteration is negative, it show not more sensitive at initial points and convergent to fixed- point. Same results for $0 < r \leq 12.3053$, for $11.8311 \leq r \leq 12.9692$, it shows cyclic behavior, see the behavior of the map in Figure 8, 9, 10, 11. , and see the behavior of the map in Figure 17, 18, 19, 20.

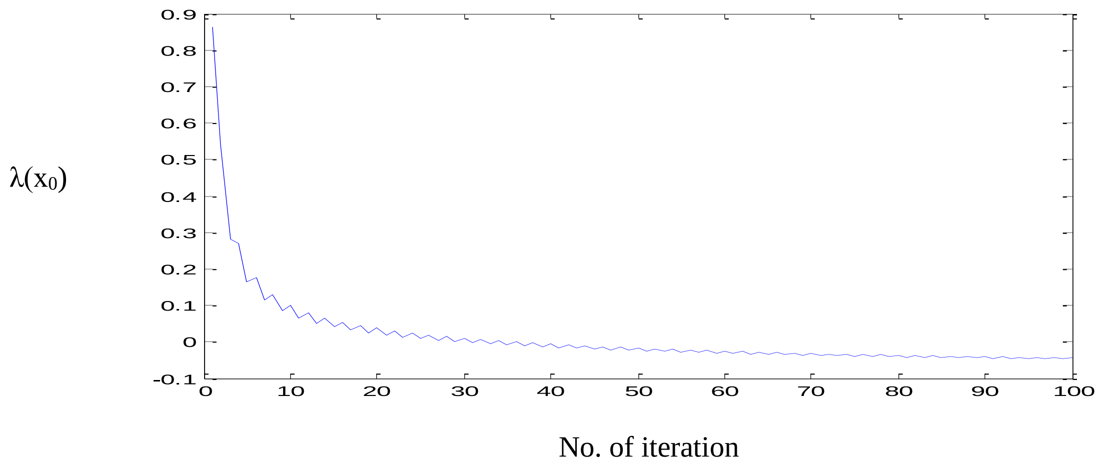


Figure 5. ($\beta = 0.1$, r = 12.9599, x_0= 0.15)

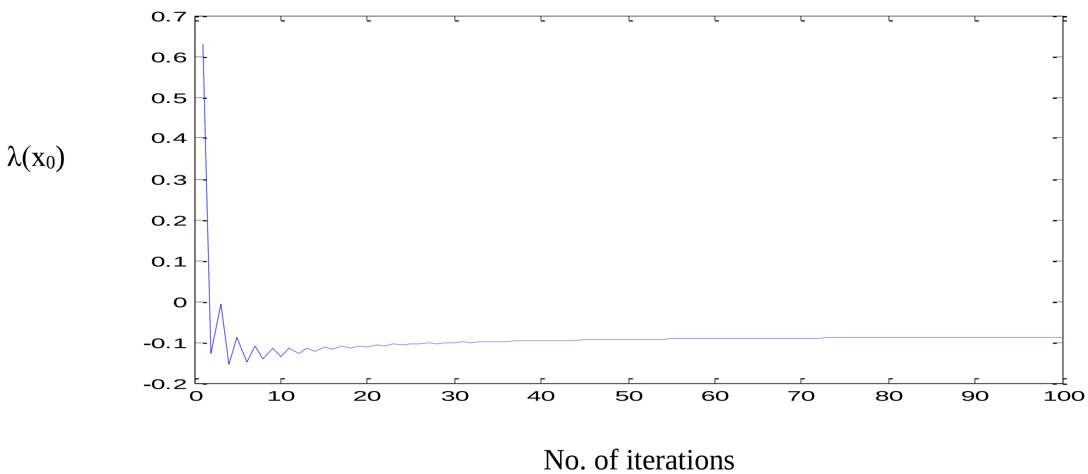


Figure 6. ($\beta = 0.1$, r = 12.3053, x_0= 0.25)

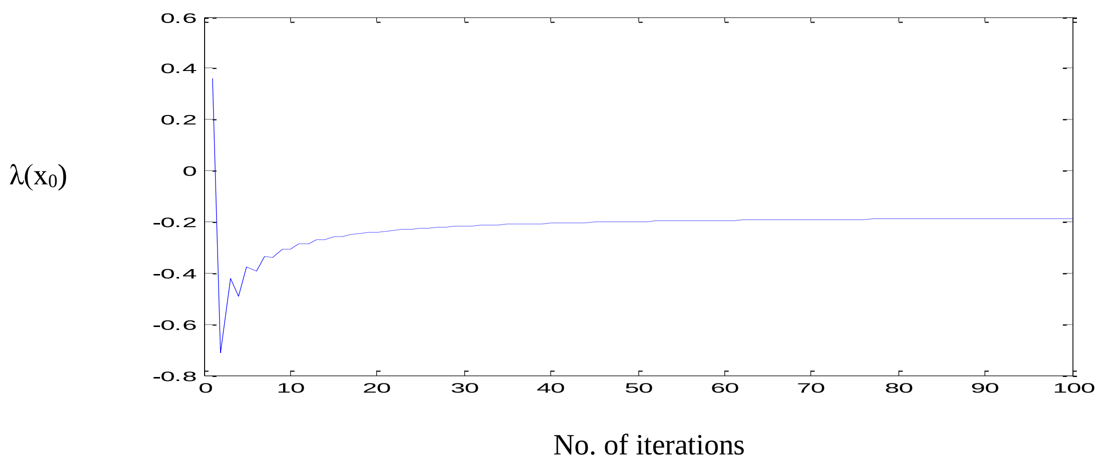


Figure 7. ($\beta = 0.17$, r = 11.8311, x_0= 0.35)

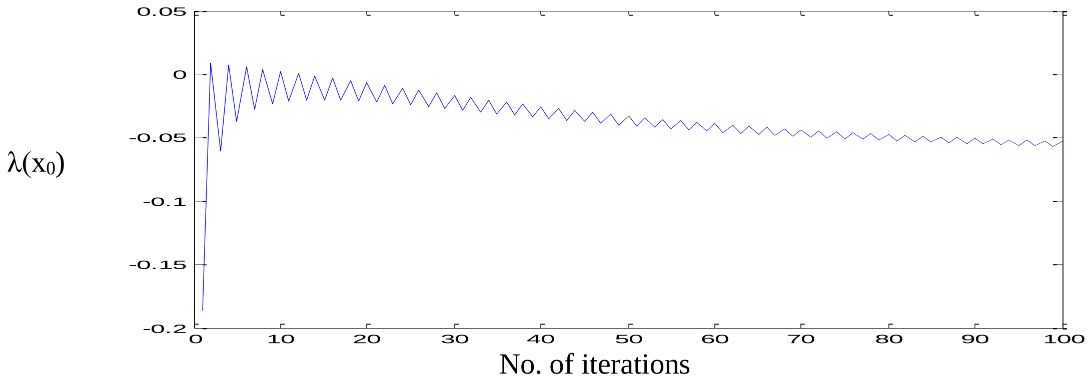


Figure 8. (β = 0.17, r = 12.9692, x_0= 0.5)

Table 8:

$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \ln\lvert\beta r(1-2x_k)+(1-\beta)\rvert$, β = 0.1, r = 18.9000				
iteration	x_0=0.15	x_0=0.25	x_0=0.35	x_0=0.5
10	-0.2384	-0.1760	-0.5298	-0.2325
100	-0.2365	-0.2296	-0.2649	-0.2353
1000	-0.2358	-0.2351	-0.2386	-0.2357
8000	-0.2357	-0.2356	-0.2361	-0.2357
9000	-0.2357	-0.2357	-0.2360	-0.2357
10000	-0.2357	-0.2357	-0.2360	-0.2357

In Table 8, for $0 < r \leq 18.9000$, for all x belonging to [0, 1]. Lyapunov exponent $\lambda(x_0)$ of logistic map via mann iteration is negative, it show cyclic behavior and not more sensitive at initial points and convergent to fixed- point. See the behavior of the map in Figure 9,10, 11, 12 and see the behavior of the map in Figure 21, 22, 23, 24.

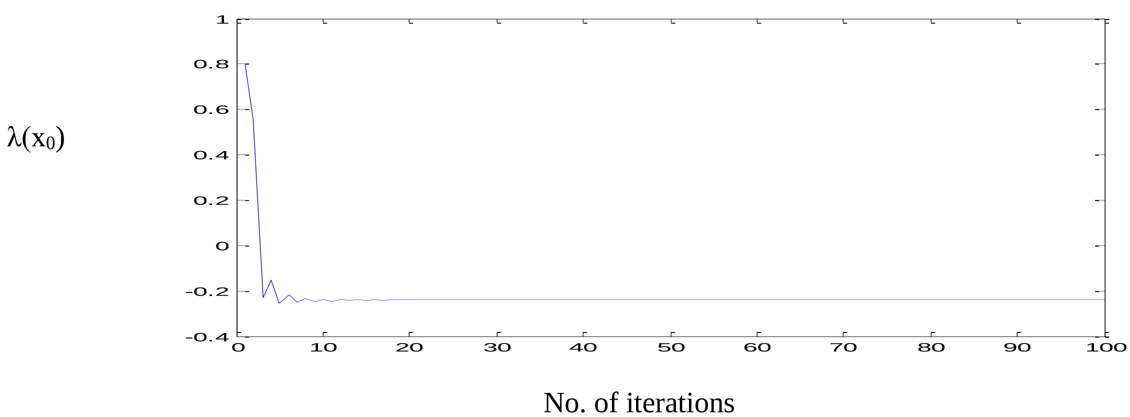


Figure 9. (β = 0.1, r = 18.9000, x_0= 0.15)

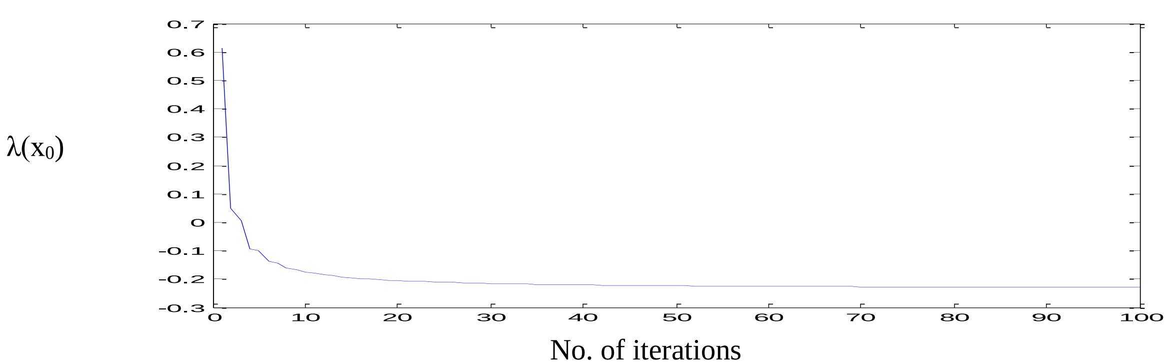


Figure 10. (β = 0.1, r = 18.9000, x_0= 0.25)

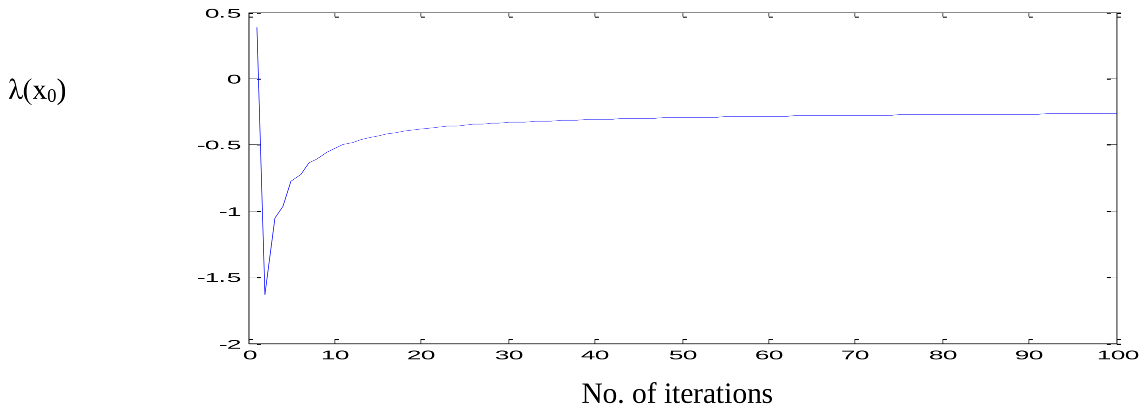


Figure 11. (β = 0.1, r = 18.9000, x_0= 0.35)

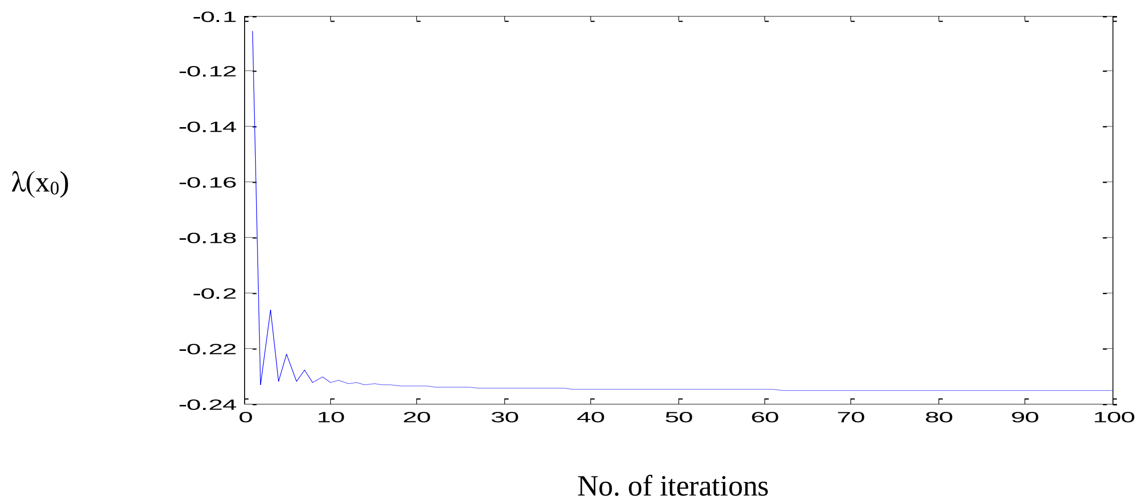


Figure 12. (β = 0.1, r = 18.9000, x_0= 0.5)

We notice that for r > 5.0015, r > 18.9000, and for larger value of r, than given in Table 6 and Table 7, logistic map cannot be described as superior orbit x_n is greater than 1, i.e. $x_n \notin [0, 1]$. See [15],

Figure 13 to 24 is mapped with results $\frac{1}{n}\sum_{k=1}^{n} \ln|{E_k}/{E_0}| \cong \lambda(x_0)$, from Table 1 and Table 2.

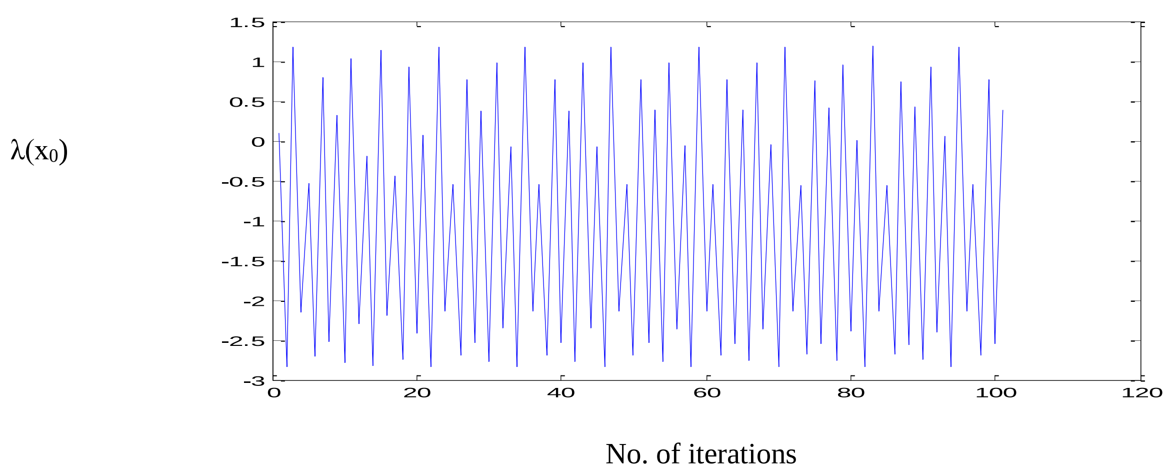
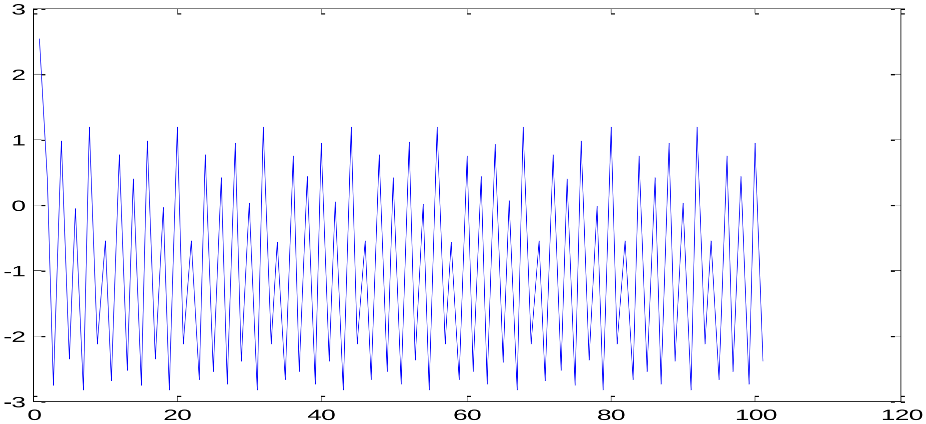


Figure 13. (β = 0.9, r = 3.8705, x_0= 0.5)

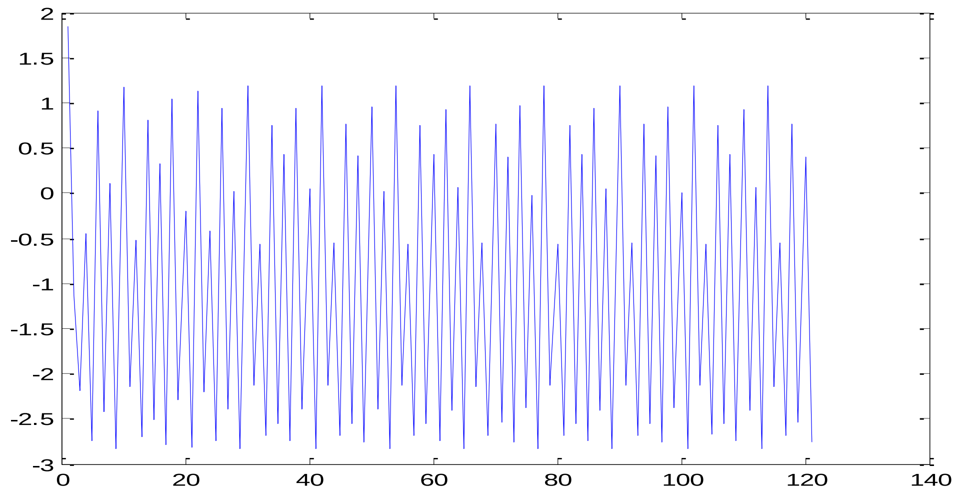
$\lambda(x_0)$



No. of iterations

Figure 14. ($\beta = 0.9$, $r = 3.8705$, $x_0 = 0.15$)

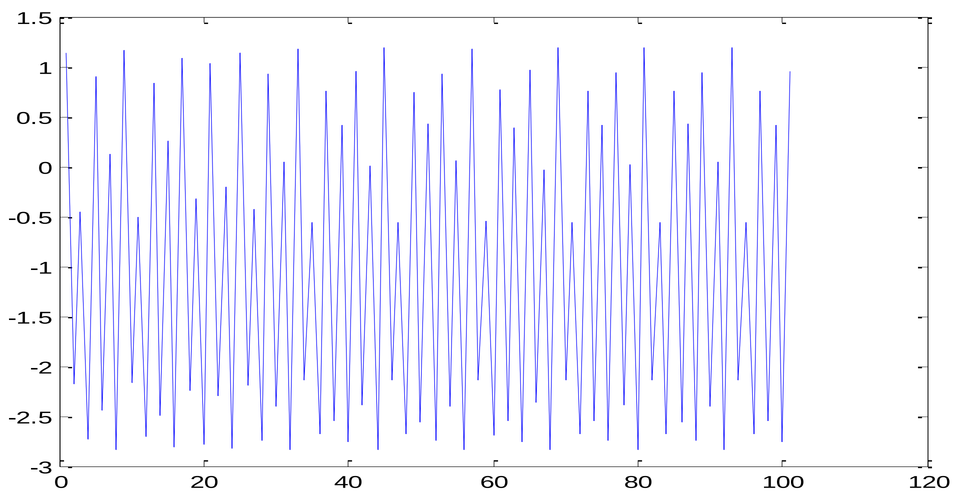
$\lambda(x_0)$



No. of iterations

Figure 15. ($\beta = 0.9$, $r = 3.8705$, $x_0 = 0.35$)

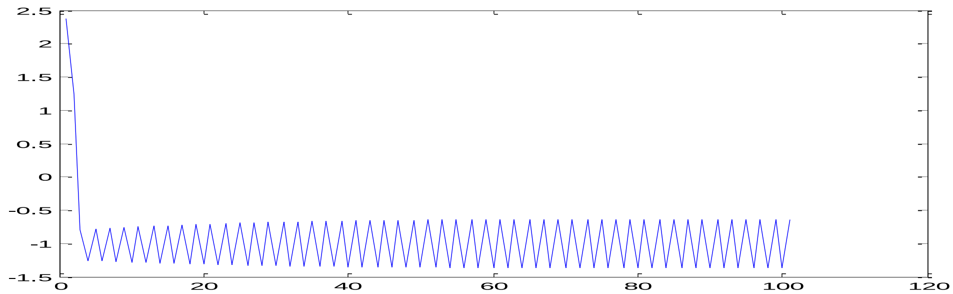
$\lambda(x_0)$



No. of iterations

Figure 16. ($\beta = 0.9$, $r = 3.8705$, $x_0 = 0.35$)

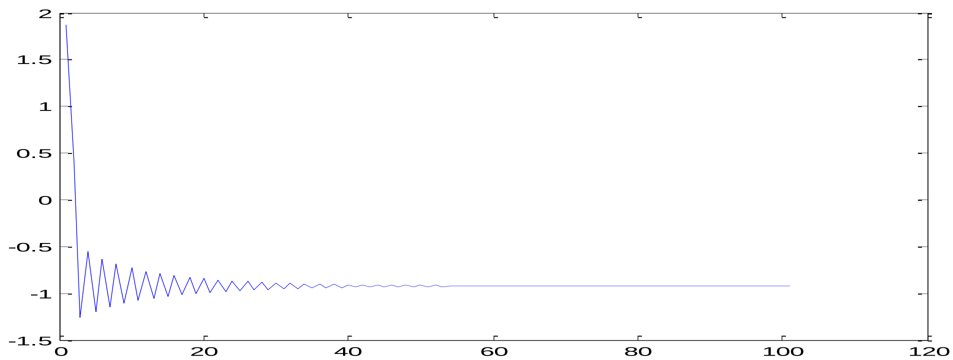
$\lambda(x_0)$



No. of iterations

Figure 17. ($\beta = 0.17$, r = 12.9599, $x_0 = 0.15$)

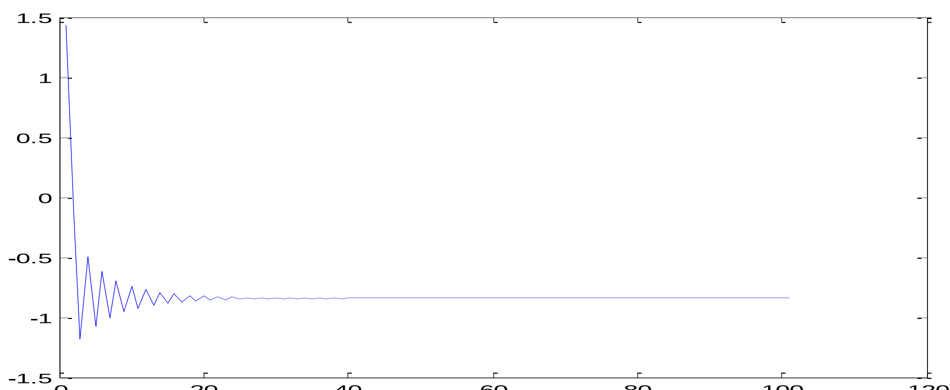
$\lambda(x_0)$



No. of iterations

Figure 18. ($\beta = 0.17$, r = 12.3053, $x_0 = 0.25$)

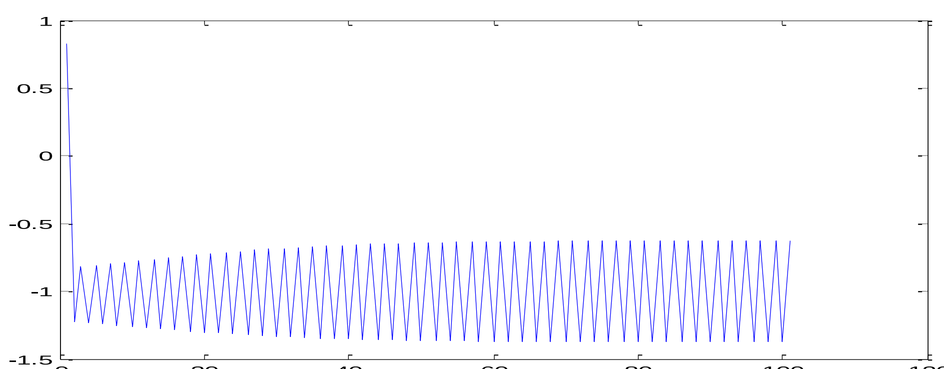
$\lambda(x_0)$



No. of iterations

Figure 19. ($\beta = 0.9$, r = 11.8311, $x_0 = 0.35$)

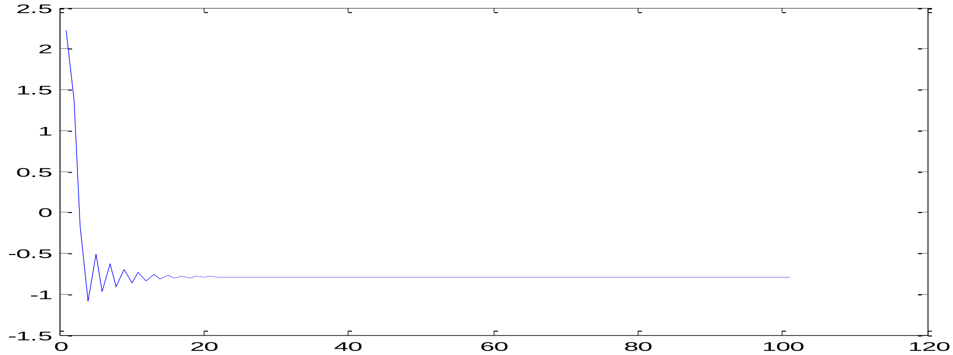
$\lambda(x_0)$



No. of iterations

Figure 20. ($\beta = 0.17$, r = 12.9692, $x_0 = 0.5$)

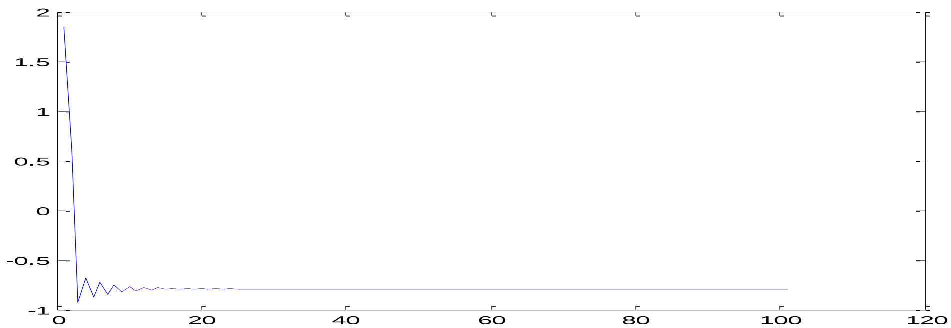
$\lambda(x_0)$



No. of iterations

Figure 21. ($\beta = 0.1$, $r = 18.9$, $x_0 = 0.15$)

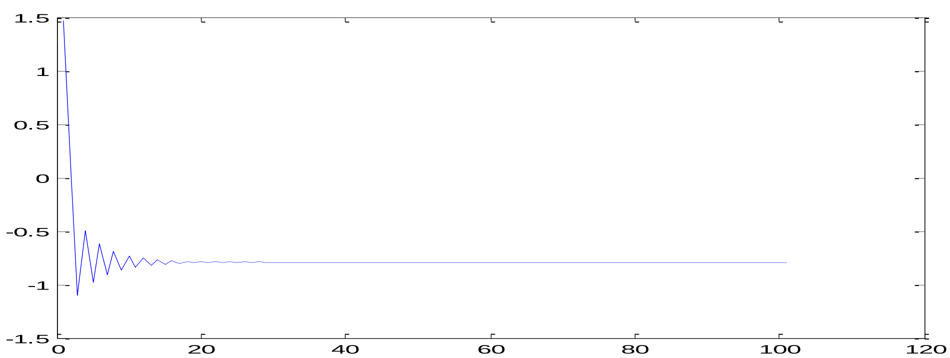
$\lambda(x_0)$



No. of iterations

Figure 22. ($\beta = 0.1$, $r = 18.9$, $x_0 = 0.25$)

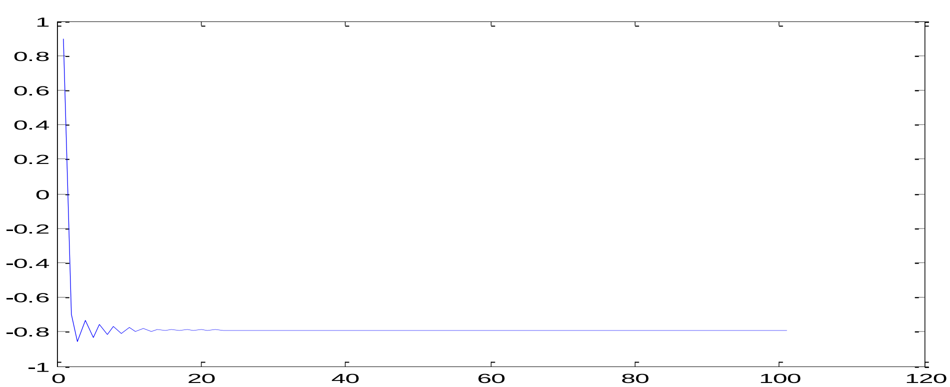
$\lambda(x_0)$



No. of iterations

Figure 23. ($\beta = 0.1$, $r = 18.9$, $x_0 = 0.35$)

$\lambda(x_0)$



No. of iterations

Figure 24. ($\beta = 0.1$, $r = 18.9$, $x_0 = 0.5$)

3. CONCLUSION:

On computing the Lyapunov exponent $\lambda(x_0)$ of logistic map via mann iteration.
We found the following result from our experiment:

1. The Lyapunov exponent $\lambda(x_0)$ of the logistic map is less sensitive for higher value of r with mann iterates.
2. Lyapunov exponent $\lambda(x_0)$ is less sensitive as the value of β comes closer to 0 logistic maps shown more sensitivity, it mean unstable behavior, i.e. it is either convergent or cyclic.
3. For $\beta < 0.17$, the logistic map is convergent to fixed point
4. For $\beta < 0.64$ logistic map show chaotic behavior.

We notice that when we are compare the conclusion of this paper with paper [15], both conclusion are same in different way. Hence we can say that the conclusion (stability and instability of paper [15] have found due to the cause of errors and sensitivity which shown in calculation of Lyapunov exponent $\lambda(x_0)$, i.e. we found the causes with the help of Lyapunov exponent $\lambda(x_0)$ calculation.

These all calculation and conclusion is between 0 and 1, if we increase or decrease limitation of x or β, then raise the questions. What about attractors? Where is the boundary conditions? Suppose that the dynamical system is dissipative and that it admits an attractor with an open basin of attraction. Since the dynamics inside the basin are dissipative, every invariant measure supported inside the basin must be supported on the attractor (zero away from the attractor) and must be singular with respect to Lebesgue measure (supported on a set of Lebesgue measure zero).

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