Algebra of Concrete Matrices and its Properties

Jayanta Biswas
Assistant Professor, Department of Mathematics, Barasat Government College,
Barasat, North 24 Parganas, West Bengal, India.
Email - jbiswas37@gmail.com

Abstract: *Here we introduce concrete matrices and algebra of them, i.e., 'addition' and 'multiplication' of concrete matrices. The extended matrix algebra[1] $(M(F),+,.)$ is a weak hemi-ring with zero $O_{1\times1}=(0)_{1\times1}\in M(F)$, where $M(F)$ is the set of all matrices over a given field F, but not a ring and so we lose many properties of ring.*
Observing this fact and following [1 – 5] we get motivation to this type of matrix algebra (matrix addition and matrix multiplication) so that the algebraic structure forms a ring. Finally, we shall study some properties of this ring.

Key Words: *Concrete Matrix, Concretization of Matrices, concretized matrix.*

Some Notations : (i) $M_{m\times n}(F)$ denotes the set of all $m\times n$ matrices over a given field F.
(ii) $A_{m\times n}\in M(F)$ denotes $A_{m\times n}$ is an $m\times n$ matrix in $M(F)$.
(iii) $O_{m\times n}$ denotes the $m\times n$ matrix in $M(F)$, of which all the elements are zero.
(iv) If $A_{m\times n}=(a_{ij})_{m\times n}\in M(F)$ and p, q are positive integers such that $p\le m,\ q\le n$, then $A_{p\times q}=(a_{ij})_{p\times q}$.

1. INTRODUCTION : As per [1],

Definition (0.1) The addition on $M(F)$ is defined by, for all $A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{p\times q}\in M(F)$,
$A+B=(c_{ij})_{r\times s}$, where $r=\max\{m,p\}$, $s=\max\{n,q\}$ and for $i=1,2,\ldots,r$; $j=1,2,\ldots,s$,
$c_{ij}=a'_{ij}+b'_{ij}$, where $a'_{ij}=\begin{cases}a_{ij}, & if\ 1\le i\le m,\ 1\le j\le n\\ 0, & otherwise\end{cases}$, for $i=1,2,\ldots,r$; $j=1,2,\ldots,s$ and $b'_{ij}=\begin{cases}b_{ij}, & if\ 1\le i\le p,\ 1\le j\le q\\ 0, & otherwise\end{cases}$, for $i=1,2,\ldots,r$; $j=1,2,\ldots,s$.

Definition (0.2) The multiplication on $M(F)$ is defined by, for all $A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{p\times q}\in M(F)$,
$AB=(c_{ij})_{m\times q}$, where for $i=1,2,\ldots,m$; $j=1,2,\ldots,q$, $c_{ij}=\sum_{k=1}^{\min\{n,p\}}a_{ik}b_{kj}$.

Then $(M(F),+,.)$ is a weak hemi-ring with zero $O_{1\times1}=(0)_{1\times1}\in M(F)$.

Definition (0.3) A weak hemi-ring is an algebraic structure $(H,+,.)$ with two binary operations + and ' . ', respectively called, addition and multiplication, such that $(H,+)$ is a commutative monoid with identity *0* (say), called zero; $(H,.)$ is a semi-group; multiplication is distributive over addition and $a.0\neq0, 0.a\neq0,\forall\ a\in H$, in general.

Definition (0.4) For $, n\in\mathbb{N}$, $I_{m\times n}=(\delta_{ij})_{m\times n}$, where for $i=1,2,\ldots,m$; $j=1,2,\ldots,n$
$\delta_{ij}=\begin{cases}1, & if\ \ i=j\\ 0, & if\ \ i\neq j\end{cases}$

Theorem (0.1) Let $\le n$. Then for two non-zero matrices $A_{m\times n}, B_{m\times n}\in M(F)$, $A_{m\times n}B_{m\times n}=I_{m\times n}$ iff $A_{m\times m}B_{m\times m}=B_{m\times m}A_{m\times m}=I_m$ and for $j=m+1,m+2,\ldots,n$, each j^{th} column B_j (say) of $B_{m\times n}$ is zero.

Corollary (0.1) Let $n\le m$. Then for two non-zero matrices $A_{m\times n}, B_{m\times n}\in M(F)$, $A_{m\times n}B_{m\times n}=I_{m\times n}$ iff $A_{n\times n}B_{n\times n}=B_{n\times n}A_{n\times n}=I_n$ and for $i=n+1,n+2,\ldots,m$, each i^{th} row A_i (say) of $A_{m\times n}$ is zero.

Theorem (0.2) For all $A_{m\times n}, B_{m\times n}\in M(F)$,
(i) $(A_{m\times n}+B_{m\times n})^T=A^T_{m\times n}+B^T_{p\times q}$ (ii) $(A_{m\times n}B_{m\times n})^T=B^T_{p\times q}A^T_{m\times n}$.

Main Results :
Since $(M(F), +, .)$ is a weak hemi-ring we lose many properties of traditional matrix algebra. In fact algebra of concrete matrices is nothing but squeezing of $(M(F), +, .)$ to give some concreteness and get a ring structure.

Definition (1.1) For all $A, B \in M(F)$, $A - B = A + (-1)B$[1].

Define a binary relation ρ on $M(F)$ by
$\rho = \{(A_{m\times n}, B_{p\times q}) \in M(F) \times M(F) : A_{m\times n} - B_{p\times q} = O_{r\times s}, r = \max\{m, p\}, s = max\{n, q\}\}$. Then ρ is an equivalence relation on $M(F)$ and so we have the quotient set $M(F)/\rho$. Let us denote this quotient set by $M_\rho(F)$.

Definition (1.2) : Define 'addition' and 'multiplication' on $M_\rho(F)$ by $\forall$ $[A], [B] \in M_\rho(F)$, $[A] + [B] = [A + B]$ and $[A][B] = [AB]$, where $A + B$ and AB are defined as per [1], and $[A]$ denotes the ρ −equivalence class of $A \in M(F)$.

Note (1.1) It can be easily checked that 'Addition' and 'Multiplication' on $M_\rho(F)$, as defined in definition(1.2) are well-defined.

Theorem (1.1) $(M_\rho(F), +, .)$ is a non-commutative ring without unity.
Proof : Clearly $M_\rho(F)$ is closed with respect to 'addition' and 'multiplication'.
Let $[A_{m\times n}]$, $[B_{p\times q}]$, $[C_{r\times s}] \in M_\rho(F)$ be arbitrary.
Now, $[A_{m\times n}] + [B_{p\times q}] = [A_{m\times n} + B_{p\times q}] = [B_{p\times q} + A_{m\times n}]^{[1]} = [B_{p\times q}] + [A_{m\times n}]$ and so 'addition' is commutative.
Again, $([A_{m\times n}] + [B_{p\times q}]) + [C_{r\times s}] = [A_{m\times n} + B_{p\times q}] + [C_{r\times s}] = [(A_{m\times n} + B_{p\times q}) + C_{r\times s}]$
$= [A_{m\times n} + (B_{p\times q} + C_{r\times s})]^{[1]} = [A_{m\times n}] + [B_{p\times q} + C_{r\times s}] = [A_{m\times n}] + ([B_{p\times q}] + [C_{r\times s}])$.
Therefore 'addition' is associative.

We see that $[O_{1\times 1}] \in M_\rho(F)$, and $[A_{m\times n}] + [O_{1\times 1}] = [A_{m\times n} + O_{1\times 1}] = [A_{m\times n}]^{[1]}$. Therefore $[O_{1\times 1}]$ is additive identity in $M_\rho(F)$.
Now $[-A_{m\times n}] \in M_\rho(F)$, and $[A_{m\times n}] + [-A_{m\times n}] = [O_{1\times 1}]$. Therefore $[-A_{m\times n}]$ is additive inverse of $[A_{m\times n}]$ in $M_\rho(F)$.
Again $([A_{m\times n}][B_{p\times q}])[C_{r\times s}] = [A_{m\times n}B_{p\times q}][C_{r\times s}] = [(A_{m\times n}B_{p\times q})C_{r\times s}]$
$= [A_{m\times n}(B_{p\times q}C_{r\times s})]^{[1]} = [A_{m\times n}][B_{p\times q}C_{r\times s}] = [A_{m\times n}]([B_{p\times q}][C_{r\times s}])$.
Therefore 'multiplication' is associative.

Also, $[A_{m\times n}]([B_{p\times q}] + [C_{r\times s}]) = [A_{m\times n}][B_{p\times q} + C_{r\times s}] = [A_{m\times n}(B_{p\times q} + C_{r\times s})]$
$= [A_{m\times n}B_{p\times q} + A_{m\times n}C_{r\times s}]^{[1]} = [A_{m\times n}B_{p\times q}] + [A_{m\times n}C_{r\times s}] = [A_{m\times n}][B_{p\times q}] + [A_{m\times n}][C_{r\times s}]$.
Therefore 'multiplication' is left distributive over 'addition'.

And $([B_{p\times q}] + [C_{r\times s}])[A_{m\times n}] = [B_{p\times q} + C_{r\times s}][A_{m\times n}] = [(B_{p\times q} + C_{r\times s})A_{m\times n}]$
$= [B_{p\times q}A_{m\times n} + C_{r\times s}A_{m\times n}]^{[1]} = [B_{p\times q}A_{m\times n}] + [C_{r\times s}A_{m\times n}] = [B_{p\times q}][A_{m\times n}] + [C_{r\times s}][A_{m\times n}]$.
Therefore 'multiplication' is right distributive over 'addition'.

Hence $(M_\rho(F), +, .)$ is a ring.

Since extended matrix multiplication is not commutative in $M(F)$, it is clear that multiplication on $M_\rho(F)$ is not commutative.
Since product of two matrices of given orders in $M(F)$[1] is, in general, a matrix of order different from the given orders, hence the ring $(M_\rho(F), +, .)$ has no unity.

2. CONCRETE MATRIX ALGEBRA :

Definition (1.3) A non-zero matrix $A_{m\times n} \in M(F)$ is said to be a concrete matrix if the m^{th} row of $A_{m\times n}$ is a non-zero row and the n^{th} column of $A_{m\times n}$ is a non-zero column. The only concrete zero matrix in $M(F)$ is $O_{1\times 1}$.

Concretization of Matrices :

Given any $A_{m\times n} \in M(F)$, we can obtain the concrete matrix ${A^c}_{m\times n}$ from $A_{m\times n}$ as follows :
If $A_{m\times n}$ be a concrete matrix then ${A^c}_{m\times n} = A_{m\times n}$.
If $A_{m\times n}$ be not a concrete matrix then discard all the zero rows and zero columns only from $A_{m\times n}$, starting from the last row and last column until we get concrete matrix, and denote this concrete matrix by ${A^c}_{m\times n}$.
This procedure is called concretization of a matrix $A_{m\times n} \in M(F)$ and the resultant concrete matrix ${A^c}_{m\times n}$ is called the concretized matrix of $A_{m\times n}$.

Theorem (1.2)(i) For any $A_{m\times n} \in M(F)$, $[A_{m\times n}] = [{A^c}_{m\times n}]$
(ii) Let $[A_{m\times n}] \in M_\rho(F)$ and $A_{m\times n} \in CM(F)$, the set of all concrete matrices over the field F.
For all $B_{p\times q} \in CM(F)$, $B_{p\times q} \in [A_{m\times n}]$ iff $B_{p\times q} = A_{m\times n}$.
Proof : Trivial.

Definition (1.4) Let $CM(F)$ be the set of all concrete matrices over a given field F . Define two binary operations $\oplus$, $\odot$ on $CM(F)$, called 'addition' and 'multiplication' of concrete matrices respectively, as follows: $\forall\, A_{m\times n}, B_{p\times q} \in CM(F)$, $A_{m\times n} \oplus B_{p\times q} = \left(A_{m\times n} + B_{p\times q}\right)^c$, the concretized matrix of $A_{m\times n} + B_{p\times q}$; and $A_{m\times n} + B_{p\times q}$ is obtained as per [1]. And , $A_{m\times n} \odot B_{p\times q} = \left(A_{m\times n}B_{p\times q}\right)^c$, the concretized matrix of $A_{m\times n}B_{p\times q}$; and $A_{m\times n}B_{p\times q}$ is obtained as per [1].

Theorem (1.3) For all $A_{m\times n}, B_{p\times q}, C_{r\times s} \in CM(F)$, $\left(A_{m\times n} \oplus B_{p\times q}\right) \oplus C_{r\times s} = \left(A_{m\times n} + B_{p\times q} + C_{r\times s}\right)^c$.
Proof : We have $A_{m\times n} \oplus B_{p\times q} = \left(A_{m\times n} + B_{p\times q}\right)^c \in \left[\left(A_{m\times n} + B_{p\times q}\right)^c\right] = \left[A_{m\times n} + B_{p\times q}\right]$ ....(1)
(by theorem(1.2)(i)).
Therefore $\left(A_{m\times n} \oplus B_{p\times q}\right) \oplus C_{r\times s} \in \left[\left(A_{m\times n} \oplus B_{p\times q}\right) + C_{r\times s}\right]$ ....(2)
(by (1), replacing $A_{m\times n}$ by $A_{m\times n} \oplus B_{p\times q}$ and $B_{p\times q}$ by $C_{r\times s}$)
Now $\left[\left(A_{m\times n} \oplus B_{p\times q}\right) + C_{r\times s}\right] = \left[A_{m\times n} \oplus B_{p\times q}\right] + [C_{r\times s}] = \left[A_{m\times n} + B_{p\times q}\right] + [C_{r\times s}]$ (by (1))
$= \left[\left(A_{m\times n} + B_{p\times q}\right) + C_{r\times s}\right] = \left[A_{m\times n} + B_{p\times q} + C_{r\times s}\right]^{[1]}$ ....(3)
From (2) and (3) we get $\left(A_{m\times n} \oplus B_{p\times q}\right) \oplus C_{r\times s} \in \left[A_{m\times n} + B_{p\times q} + C_{r\times s}\right]$.

Now $\left(A_{m\times n} \oplus B_{p\times q}\right) \oplus C_{r\times s}$ and $\left(A_{m\times n} + B_{p\times q} + C_{r\times s}\right)^c$ both are concrete matrices in $\left[A_{m\times n} + B_{p\times q} + C_{r\times s}\right]$ (by theorem(1.2)(i)) and so by theorem(1.2)(ii) we can say that $\left(A_{m\times n} \oplus B_{p\times q}\right) \oplus C_{r\times s} = \left(A_{m\times n} + B_{p\times q} + C_{r\times s}\right)^c$. This completes the proof.

Theorem (1.4) For all $A_{m\times n}, B_{p\times q}, C_{r\times s} \in CM(F)$, $A_{m\times n} \oplus \left(B_{p\times q} \oplus C_{r\times s}\right) = \left(A_{m\times n} + B_{p\times q} + C_{r\times s}\right)^c$.
Proof : Almost similar to proof of theorem(1.3).

Theorem (1.5) For all $A_{m\times n}, B_{p\times q}, C_{r\times s} \in CM(F)$,
(i) $\left(A_{m\times n} \odot B_{p\times q}\right) \odot C_{r\times s} = \left(A_{m\times n}B_{p\times q}C_{r\times s}\right)^c$. **(ii)** $A_{m\times n} \odot \left(B_{p\times q} \odot C_{r\times s}\right) = \left(A_{m\times n}B_{p\times q}C_{r\times s}\right)^c$
Proof : Almost similar to proof of theorem(1.3).

Theorem (1.6) For all $A_{m\times n}, B_{p\times q}, C_{r\times s} \in CM(F)$,
(i) $\left[A_{m\times n}\left(B_{p\times q} + C_{r\times s}\right)^c\right] = \left[A_{m\times n}\left(B_{p\times q} + C_{r\times s}\right)\right]$
(ii) $\left[\left(A_{m\times n} B_{p\times q}\right)^c + \left(A_{m\times n}C_{r\times s}\right)^c\right] = \left[A_{m\times n} B_{p\times q} + A_{m\times n}C_{r\times s}\right]$
(iii) $\left[\left(B_{p\times q} + C_{r\times s}\right)^c A_{m\times n}\right] = \left[\left(B_{p\times q} + C_{r\times s}\right)A_{m\times n}\right]$
(iv) $\left[\left(B_{p\times q}A_{m\times n}\right)^c + \left(C_{r\times s}A_{m\times n}\right)^c\right] = \left[B_{p\times q}A_{m\times n} + C_{r\times s}A_{m\times n}\right]$
Proof : (i) Since $\left(B_{p\times q} + C_{r\times s}\right) - \left(B_{p\times q} + C_{r\times s}\right)^c = O_{u\times v}$, where $u = \max\{p, r\}$, $v = \max\{q, s\}$,
hence $A_{m\times n}\left(B_{p\times q} + C_{r\times s}\right) - A_{m\times n}\left(B_{p\times q} + C_{r\times s}\right)^c = A_{m\times n}\left(\left(B_{p\times q} + C_{r\times s}\right) - \left(B_{p\times q} + C_{r\times s}\right)^c\right) = O_{m\times v}$.

Hence the result. Similarly we can prove (ii), (iii) and (iv).

Theorem (1.7) $(CM(F), \oplus, \odot)$ is a non-commutative ring without unity.
Proof : From definition(1.4) it is clear that $CM(F)$ is closed with respect to both $\oplus$, $\odot$.

Let $A_{m\times n}, B_{p\times q}, C_{r\times s} \in CM(F)$ be arbitrary.
Now $A_{m\times n} \oplus B_{p\times q} = (A_{m\times n} + B_{p\times q})^c = (B_{p\times q} + A_{m\times n})^c$ (as per [1])
$= B_{p\times q} \oplus A_{m\times n}$.
Therefore $\oplus$ is commutative.

From theorem (1.3) and theorem(1.4) we have $(A_{m\times n} \oplus B_{p\times q}) \oplus C_{r\times s} = A_{m\times n} \oplus (B_{p\times q} \oplus C_{r\times s})$
Therefore $\oplus$ is associative.

Now $O_{1\times 1} \in CM(F)$ and $A_{m\times n} \oplus O_{1\times 1} = (A_{m\times n} + O_{1\times 1})^c = A^c_{m\times n} = A_{m\times n}$ (since $A_{m\times n} \in CM(F)$).
Therefore $O_{1\times 1}$ is additive identity in (F) .

Since $A_{m\times n} \in CM(F)$, hence $-A_{m\times n} \in CM(F)$, and
$A_{m\times n} \oplus (-A_{m\times n}) = (A_{m\times n} + (-A_{m\times n}))^c = O^c_{m\times n} = O_{1\times 1}$.
Therefore every element of $CM(F)$ has additive inverse in $CM(F)$.

From theorem(1.5)(i) and theorem(1.5)(ii) we have $(A_{m\times n} \odot B_{p\times q}) \odot C_{r\times s} = A_{m\times n} \odot (B_{p\times q} \odot C_{r\times s})$.
Therefore $\odot$ is associative.
Now,

$A_{m\times n} \odot (B_{p\times q} \oplus C_{r\times s}) = \left(A_{m\times n}(B_{p\times q} + C_{r\times s})^c\right)^c \in \left[A_{m\times n}(B_{p\times q} + C_{r\times s})^c\right] = [A_{m\times n}(B_{p\times q} + C_{r\times s})]$
(By Theorem(1.6)(i)).

But $[A_{m\times n}(B_{p\times q} + C_{r\times s})] = [A_{m\times n}B_{p\times q} + A_{m\times n}C_{r\times s}]$
(since $A_{m\times n}(B_{p\times q} + C_{r\times s}) = A_{m\times n}B_{p\times q} + A_{m\times n}C_{r\times s}$)[1]

Therefore $A_{m\times n} \odot (B_{p\times q} \oplus C_{r\times s})$ is a concrete matrix in $[A_{m\times n}B_{p\times q} + A_{m\times n}C_{r\times s}]$ .... (1)

Again, $(A_{m\times n} \odot B_{p\times q}) \oplus (A_{m\times n} \odot C_{r\times s}) = \left((A_{m\times n}B_{p\times q})^c + (A_{m\times n}C_{r\times s})^c\right)^c \in$
$\left[(A_{m\times n}B_{p\times q})^c + (A_{m\times n}C_{r\times s})^c\right] = [A_{m\times n} B_{p\times q} + A_{m\times n}C_{r\times s}]$ (By Theorem(1.6)(ii)).

Therefore $(A_{m\times n} \odot B_{p\times q}) \oplus (A_{m\times n} \odot C_{r\times s})$ is a concrete matrix in $[A_{m\times n}B_{p\times q} + A_{m\times n}C_{r\times s}]$ .... (2)
Therefore, from (1) and (2) and by Theorem(1.2)(ii), we have
$A_{m\times n} \odot (B_{p\times q} \oplus C_{r\times s}) = (A_{m\times n} \odot B_{p\times q}) \oplus (A_{m\times n} \odot C_{r\times s})$.
Therefore left distributive property holds.

Similarly, by Theorem(1.6)(iii), Theorem(1.6)(iv) and Theorem(1.2)(ii) we can prove the right distributive property.
Therefore $(CM(F), \oplus, \odot)$ is a ring.

Since extended matrix multiplication is not commutative in $M(F)$, it is clear that multiplication on $CM(F)$ is not commutative.

Since product of two matrices of given orders in $M(F)$ is, in general, a matrix of order different from the given orders, hence the ring $(CM(F), \oplus, \odot)$ has no unity.

Theorem (1.8) The rings $(M_\rho(F), +, .)$ and $(CM(F), \oplus, \odot)$ are isomorphic.
Proof : Define a map $f : (CM(F), \oplus, \odot) \longrightarrow (M_\rho(F), +, .)$ by $\forall A_{m\times n} \in CM(F),\ f(A_{m\times n}) = [A_{m\times n}]$.
Let $A_{m\times n}, B_{p\times q} \in CM(F)$ be arbitrary such that $f(A_{m\times n}) = f(B_{p\times q})$, *i.e.*, $[A_{m\times n}] = [B_{p\times q}]$.
Then by Theorem(1.2)(ii) we have $A_{m\times n} = B_{p\times q}$. Therefore f is injective.

Let $[A_{m\times n}] \in M_\rho(F)$ be arbitrary. Then $A^c_{m\times n} \in CM(F)$ and $f(A^c_{m\times n}) = [A^c_{m\times n}] = [A_{m\times n}]$
(by Theorem(1.2)(i))

Therefore f is surjective.

Let $A_{m\times n}, B_{p\times q} \in CM(F)$ be arbitrary.
Then $f(A_{m\times n} \oplus B_{p\times q}) = [A_{m\times n} \oplus B_{p\times q}] = \left[(A_{m\times n} + B_{p\times q})^c\right] = [A_{m\times n} + B_{p\times q}]$ (by Theorem(1.2)(i))
$= [A_{m\times n}] + [B_{p\times q}] = f(A_{m\times n}) + f(B_{p\times q})$.

And $f\left(A_{m\times n} \odot B_{p\times q}\right) = \left[A_{m\times n} \odot B_{p\times q}\right] = \left[\left(A_{m\times n}B_{p\times q}\right)^{c}\right] = \left[A_{m\times n}B_{p\times q}\right]$ (by Theorem(1.2)(i))

$= \left[A_{m\times n}\right]\left[B_{p\times q}\right] = f\left(A_{m\times n}\right)f\left(B_{p\times q}\right).$

Therefore f is a ring isomorphism from the ring $(CM(F),\ \oplus,\ \odot)$ to the ring $\left(M_{\rho}(F),\ +,\ .\right)$.

Note (1.2) Since the rings $(CM(F),\ \oplus,\ \odot)$ and $\left(M_{\rho}(F),\ +,\ .\right)$ are isomorphic, to study about properties of these rings we shall consider any one of them whichever is suitable.
Again it is clear that for $, n \in \mathbb{N},\ \ I^{c}_{m\times n} = I_p$, where $p = \min\{m,n\},\ i.e.,\ I_{m\times n} \in CM(F)\ iff\ \ m = n.$

Now we shall study about some properties of this algebra of concrete matrices.

Definition (1.5) Let $A_{m\times n} = \left(a_{ij}\right)_{m\times n} \in CM(F)$. If $m \le n$, then for $i = 1,2,\dots,m$; $j = i, i+1,\dots, i+(n-m),\ a_{ij}'s$ are called the diagonal elements of $A_{m\times n}$.
If $m > n$, then for $j = 1,2,\dots,n$; $i = j, j+1,\dots, j+(m-n),\ a_{ij}'s$ are called the diagonal elements of $A_{m\times n}$ .In each case, the portion of $A_{m\times n}$, formed by these diagonal elements is called the diagonal of $A_{m\times n}$.
All the elements of $A_{m\times n}$, other than the diagonal elements, are called the non-diagonal elements of $A_{m\times n}$.

Definition (1.6) **(a)** A matrix $A_{m\times n} = \left(a_{ij}\right)_{m\times n} \in CM(F)$ is said to be symmetric if, when $m \le n$, then for $i = 2,3,\dots,m$; $j = 1,2,\dots\dots,m-1$, if $i > j$ then $a_{ij} = a_{j(i+n-m)}$ and when $m \ge n$, then for $j = 2,3,\dots,n$; $i = 1,2,\dots\dots,n-1$, if $i < j$ then $a_{ij} = a_{(j+m-n)i}$.

(b) A matrix $A_{m\times n} = \left(a_{ij}\right)_{m\times n} \in CM(F)$ is said to be skew-symmetric if all the diagonal elements of $A_{m\times n}$ are zero and when $m \le n$, then for $i = 2,3,\dots,m$; $j = 1,2,\dots\dots,m-1$, if $i > j$ then
$a_{ij} = -a_{j(i+n-m)}$ and when $m \ge n$, then for $j = 2,3,\dots,n$; $i = 1,2,\dots\dots,n-1$, if $i < j$ then
$a_{ij} = -a_{(j+m-n)i}$.

(c) A matrix $A_{m\times n} = \left(a_{ij}\right)_{m\times n} \in CM(F)$ is said to be weak skew-symmetric if, when $m \le n$, then for $i = 2,3,\dots,m$; $j = 1,2,\dots\dots,m-1$, if $i > j$ then $a_{ij} = -a_{j(i+n-m)}$ and when $m \ge n$, then for
$j = 2,3,\dots,n$; $i = 1,2,\dots\dots,n-1$, if $i < j$ then $a_{ij} = -a_{(j+m-n)i}$.

Example (1.1) Among the concrete real matrices $A = \begin{pmatrix} \mathbf{2} & \mathbf{1} & 0 & -1 \\ 0 & \mathbf{5} & \mathbf{2} & 3 \\ -1 & 3 & \mathbf{-5} & \mathbf{7} \end{pmatrix}, B = \begin{pmatrix} \mathbf{0} & 1 & -3 \\ \mathbf{0} & \mathbf{0} & 4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -1 & \mathbf{0} & \mathbf{0} \\ 3 & -4 & \mathbf{0} \end{pmatrix},$

$C = \begin{pmatrix} \mathbf{2} & 1 & -3 \\ \mathbf{0} & \mathbf{5} & 4 \\ \mathbf{1} & \mathbf{0} & \mathbf{8} \\ -1 & \mathbf{-8} & \mathbf{9} \\ 3 & -4 & \mathbf{1} \end{pmatrix}$, the bold elements are diagonal elements and the non-bold elements are non-diagonal elements.
Also A is symmetric, B is skew-symmetric and C is weak skew-symmetric.

Theorem (1.9) For two non-zero matrices $A_{m\times n}\ ,\ B_{m\times n} \in CM(F),\ \ A_{m\times n} \odot B_{m\times n} = I^{c}_{m\times n}\ iff$
$m = n$ and $A_{m\times m} \odot B_{m\times m} = B_{m\times m} \odot A_{m\times m} = I_m$.
Proof : Let $A_{m\times n} \odot B_{m\times n} = I^{c}_{m\times n}$ ....(1).
Firstly we shall show that $m = n$. If possible, let $m \ne n$.
Let $m < n$. In this case, from (1) it is clear that $A_{m\times n}B_{m\times n} = I_{m\times n}$ and so by Theorem(0.1) we have $B_{m\times n} = \left(B_{m\times m}\quad O_{m\times(n-m)}\right)$ which is not possible, since $B_{m\times n} \in CM(F)$ and $n - m \ge 1$.

Similarly, from (1) , Corollary(0.1) and from the fact that $A_{m\times n} \in CM(F)$ we have the impossibility $n < m$.
Hence $m = n$ and so (1) becomes $A_{m\times m} \odot B_{m\times m} = I_m$, $i.e.,\ \left(A_{m\times m}B_{m\times m}\right)^{c} = I_m$ ....(2).
From (2) it is clear that $\left(A_{m\times m}B_{m\times m}\right)^{c} = A_{m\times m}B_{m\times m}$ so that (2) becomes $A_{m\times m}B_{m\times m} = I_m$......(3)

From (3) we have $B_{m\times m}A_{m\times m} = I_m$ ....(4)
From(4) it is clear that $B_{m\times m}A_{m\times m} = \left(B_{m\times m}A_{m\times m}\right)^{c} = B_{m\times m} \odot A_{m\times m}$
Therefore (4) becomes $B_{m\times m} \odot A_{m\times m} = I_m$ ....(5)
From (2) and (5), we have $A_{m\times m} \odot B_{m\times m} = B_{m\times m} \odot A_{m\times m} = I_m$.

Converse part is trivial.

Theorem (1.10) For all $A_{m\times n} \in CM(F)$, $(A^c_{m\times n})^T = (A^T_{m\times n})^c$ ($A^T_{m\times n}$ is the transpose of $A_{m\times n}$).
Proof : Trivial.

Theorem (1.11) For all $A_{m\times n},\ B_{p\times q} \in CM(F)$, $\left(A_{m\times n} \oplus B_{p\times q}\right)^T = A^T_{m\times n} \oplus B^T_{p\times q}$.

Proof : $\left(A_{m\times n} \oplus B_{p\times q}\right)^T = \left(\left(A_{m\times n} + B_{p\times q}\right)^c\right)^T = \left(\left(A_{m\times n} + B_{p\times q}\right)^T\right)^c$ (by theorem(1.10))
$= \left(A^T_{m\times n} + B^T_{p\times q}\right)^c$ (By theorem(0.2)(i))
$= A^T_{m\times n} \oplus B^T_{p\times q}$

Theorem (1.12) For all $A_{m\times n},\ B_{p\times q} \in CM(F)$, $\left(A_{m\times n} \odot B_{p\times q}\right)^T = B^T_{p\times q} \odot A^T_{m\times n}$.

Proof : $\left(A_{m\times n} \odot B_{p\times q}\right)^T = \left(\left(A_{m\times n} B_{p\times q}\right)^c\right)^T = \left(\left(A_{m\times n} B_{p\times q}\right)^T\right)^c$ (by theorem(1.10))
$= \left(B^T_{p\times q} A^T_{m\times n}\right)^c$ (By theorem(0.2)(ii))
$= B^T_{p\times q} \odot A^T_{m\times n}$

Theorem (1.13) For any $A, B \in CM(F)$,
(i) if A, B be symmetric, then $A \oplus B$ may not be symmetric.
(ii) if A, B be skew-symmetric, then $A \oplus B$ may not be skew-symmetric.
(iii) if A, B be weak skew-symmetric, then $A \oplus B$ may not be weak skew-symmetric.

Proof : (i) Consider the real concrete symmetric matrices

$$A = \begin{pmatrix} \mathbf{3} & \mathbf{2} & 0 & -1 \\ 0 & \mathbf{7} & -1 & 3 \\ -1 & 3 & \mathbf{-6} & \mathbf{7} \end{pmatrix} \text{ and } B = \begin{pmatrix} \mathbf{2} & \mathbf{4} & \mathbf{3} & 4 & 5 & 6 & 7 \\ 4 & \mathbf{-1} & \mathbf{0} & \mathbf{-3} & -3 & -2 & 1 \\ 5 & -3 & \mathbf{6} & \mathbf{7} & \mathbf{3} & 4 & 5 \\ 6 & -2 & 4 & \mathbf{5} & \mathbf{2} & \mathbf{7} & 4 \\ 7 & 1 & 5 & 4 & \mathbf{3} & \mathbf{-3} & \mathbf{4} \end{pmatrix}.$$

Then $A \oplus B = \begin{pmatrix} \mathbf{5} & 6 & \mathbf{3} & 3 & 5 & 6 & 7 \\ 4 & 6 & -1 & 0 & -3 & -2 & 1 \\ 4 & 0 & 0 & \mathbf{14} & \mathbf{3} & 4 & 5 \\ 6 & -2 & 4 & 5 & \mathbf{2} & \mathbf{7} & 4 \\ 7 & 1 & 5 & 4 & \mathbf{3} & \mathbf{-3} & \mathbf{4} \end{pmatrix}$ which is not symmetric, since the (1, 4)th element of the matrix $A \oplus B$ is not equal to the (2, 1)th element.

(ii) Consider the real concrete skew-symmetric matrices

$$A = \begin{pmatrix} \mathbf{0} & \mathbf{0} & 2 & -1 \\ -2 & \mathbf{0} & \mathbf{0} & 6 \\ 1 & -6 & \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ and } B = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 4 & 5 & 6 & 7 \\ -4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -3 & -2 & 1 \\ -5 & 3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 4 & 5 \\ -6 & 2 & -4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 4 \\ -7 & -1 & -5 & -4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Then $A \oplus B = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{2} & 3 & 5 & 6 & 7 \\ -6 & \mathbf{0} & \mathbf{0} & \mathbf{6} & -3 & -2 & 1 \\ -4 & -3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 4 & 5 \\ -6 & 2 & -4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 4 \\ -7 & -1 & -5 & -4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$ which is not skew-symmetric, since the (1, 4)th entry of the matrix $A \oplus B$ is not equal to negative of the (2, 1)th element.

(iii) Since a skew-symmetric matrix is also a weak skew-symmetric matrix, the example, considered in (ii) is sufficient to establish the result.

Note (1.3) In our conventional matrix algebra we know that, if A, B be two symmetric matrices of the same order then AB is symmetric iff $AB = BA$. But in the algebra of concrete matrices, this result fails to be hold, discussed in theorem(1.14).

Theorem (1.14) If A, B be two symmetric matrices in $CM(F)$ of the same order such that $A \odot B = B \odot A$, then $A \odot B$ may not be symmetric. Again, if $A \odot B$ be symmetric then $A \odot B$ may not be equal to $B \odot A$.

Proof : For example, consider the real concrete symmetric matrices

$A = B = \begin{pmatrix} \mathbf{1} & \mathbf{2} & 3 & 4 \\ 3 & \mathbf{5} & \mathbf{6} & 7 \\ 4 & 7 & \mathbf{8} & \mathbf{9} \end{pmatrix}$. Then clearly $\odot B = B \odot A$; but $A \odot B = \begin{pmatrix} \mathbf{19} & \mathbf{33} & 39 & 45 \\ 42 & \mathbf{73} & \mathbf{87} & 101 \\ 57 & 99 & \mathbf{118} & \mathbf{137} \end{pmatrix}$ is not symmetric

Again consider the symmetric matrix $A = \begin{pmatrix} \mathbf{2} & \mathbf{1} & 0 & -1 \\ 0 & \mathbf{5} & \mathbf{2} & 3 \\ -1 & 3 & \mathbf{-5} & \mathbf{7} \end{pmatrix}$ and $B = \begin{pmatrix} \mathbf{2} & \mathbf{-26} & 1 & -1 \\ 1 & \mathbf{8} & \mathbf{1} & 8 \\ -1 & 8 & \mathbf{-1} & \mathbf{-15} \end{pmatrix}$ of the same order.

Then $A \odot B = \begin{pmatrix} \mathbf{5} & \mathbf{-44} & 3 & 6 \\ 3 & \mathbf{56} & \mathbf{3} & 10 \\ 6 & 10 & \mathbf{7} & \mathbf{100} \end{pmatrix}$ is symmetric. Now $B \odot A = \begin{pmatrix} 3 & -125 & -57 & -73 \\ 1 & 44 & 11 & 30 \\ -1 & 36 & 21 & 18 \end{pmatrix}$ so that $A \odot B \neq B \odot A$.

Theorem (1.15) We know that, for any square symmetric matrix A and any matrix P in $M(F)$, $P^T AP$ is a symmetric matrix, but the result fails to be hold good if A be a non-square concrete symmetric matrix.

Proof : Consider the real non-square concrete symmetric matrix $A = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & 4 & 5 \\ 4 & \mathbf{-1} & \mathbf{-5} & \mathbf{3} & 2 \\ 5 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{-2} \end{pmatrix}$ and another real concrete matrix $P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & -3 & 0 & -1 \\ 3 & -1 & 2 & 1 \\ -2 & 0 & 1 & 3 \end{pmatrix}$. Then $P^T \odot A \odot P = \begin{pmatrix} \mathbf{-26} & 66 & 84 & 158 \\ 67 & \mathbf{-65} & -6 & -48 \\ 62 & -15 & \mathbf{73} & 89 \\ 77 & -41 & 64 & \mathbf{66} \end{pmatrix}$ is not symmetric.

Theorem (1.16) For any matrix $A \in CM(F)$, the concrete matrix $A \oplus A^T$ is a square concrete symmetric matrix and the concrete matrix $A \oplus (-A^T)$ is a square concrete skew-symmetric matrix.
Proof : Trivial.

Theorem (1.17) Every concrete matrix over a field F can be expressed as sum of a concrete symmetric matrix and a concrete skew-symmetric matrix, but the expression is not, in general, unique, provided $char(F) \neq 2$.
Proof : Let $A = (a_{ij})_{m\times n} \in CM(F)$ be arbitrary. If $m = n$, then the result is obvious, as
$A_{m\times n} = \frac{1}{2}(A_{m\times n} \oplus (A_{m\times n})^T) \oplus \frac{1}{2}(A_{m\times n} \oplus (-(A_{m\times n})^T))$.

Let $m < n$. Let $B = (b_{ij})_{m\times n}$ be a symmetric matrix and $C = (c_{ij})_{m\times n}$ be a skew-symmetric matrix in $CM(F)$ such that $A = B \oplus C$, i.e., $(a_{ij})_{m\times n} = (b_{ij})_{m\times n} \oplus (c_{ij})_{m\times n}$ ……………… (1)
Then for $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$, $b_{ij} + c_{ij} = a_{ij}$ ……………. (2)
Since $m < n$ and B , C are symmetric and skew-symmetric matrices respectively, hence the diagonal elements of B and A are same (since the diagonal elements of C are zero). Thus the diagonal elements of B are determined.

Now for the non-diagonal elements of B and C we have
for $i = 2,3,\ldots,m$; $j = 1,2,\ldots,m-1$, if $i > j$, then $b_{ij} = b_{j(i+n-m)}$ …………… (3)
and $c_{ij} = -c_{j(i+n-m)}$ …………………….. (4)

From (2) we have, for $i = 2,3,\ldots,m$; $j = 1,2,\ldots,m-1$, if $i > j$, then $b_{ij} + c_{ij} = a_{ij}$ ……. (5)
and $b_{j(i+n-m)} + c_{j(i+n-m)} = a_{j(i+n-m)}$, i.e. , $b_{ij} - c_{ij} = a_{j(i+n-m)}$ ………………. (6) (by (3), (4)).

From (5) and (6) we get
For $= 2,3,\ldots,m$; $j = 1,2,\ldots,m-1$, if $i > j$, then $b_{ij} = 2^{-1}(a_{ij} + a_{j(i+n-m)})$ ………. (7)
$c_{ij} = 2^{-1}(a_{ij} - a_{j(i+n-m)})$ ………………. (8) , provided char(F) ≠ 2 .
$b_{j(i+n-m)} = b_{ij} = 2^{-1}(a_{ij} + a_{j(i+n-m)})$ ……….. (9) (by (7))
$c_{j(i+n-m)} = -c_{ij} = -2^{-1}(a_{ij} - a_{j(i+n-m)})$ ………….. (10) (by (8)).
From (7), (8), (9) and (10), it is clear that the last row as well as the last column of the right hand side of (1) are non-zero so that the $B \oplus C = B + C$ and hence (2) is valid.
Thus B and C are determined.

If $m > n$, then similarly, we have $\left((a_{ij})_{m\times n}\right)^T = E_{n\times m} \oplus F_{n\times m}$ ……….. (11) , where $E_{n\times m}$ is a concrete symmetric matrix and $F_{n\times m}$ is a concrete skew-symmetric matrix.
From (11), we get $A = (a_{ij})_{m\times n} = (E_{n\times m} \oplus F_{n\times m})^T = (E_{n\times m})^T \oplus (F_{n\times m})^T$ ………. (12)
(by theorem(1.11))
Since $E_{n\times m}$ is a concrete symmetric matrix and $F_{n\times m}$ is a concrete skew-symmetric matrix, hence $(E_{n\times m})^T$ is a concrete symmetric matrix and $(F_{n\times m})^T$ is a concrete skew-symmetric matrix.
Hence the result.

To establish the last part of the theorem, consider the real concrete matrix $= \begin{pmatrix} 1 & 1 & 3 & 4 \\ 4 & 2 & 1 & 3 \\ -3 & 0 & 4 & 0 \\ 2 & 1 & -2 & 0 \end{pmatrix}$.

Then $A = \frac{1}{2}(A \oplus A^T) \oplus \frac{1}{2}(A \oplus (-A^T)) = \frac{1}{2}\begin{pmatrix} \mathbf{2} & 5 & 0 & 6 \\ 5 & \mathbf{4} & 1 & 4 \\ 0 & 1 & \mathbf{8} & -2 \\ 6 & 4 & -2 & \mathbf{0} \end{pmatrix} \oplus \frac{1}{2}\begin{pmatrix} \mathbf{0} & -3 & 6 & 2 \\ 3 & \mathbf{0} & 1 & 2 \\ -6 & -1 & \mathbf{0} & 2 \\ -2 & -2 & -2 & \mathbf{0} \end{pmatrix}$ ……. (13)

and $\frac{1}{2}(A \oplus A^T)$ is concrete symmetric and $\frac{1}{2}(A \oplus (-A^T))$ is concrete skew-symmetric .

Again we see that $A = \begin{pmatrix} \mathbf{1} & 1 & 2 \\ \mathbf{5} & \mathbf{2} & 1 \\ 1 & \mathbf{3} & \mathbf{4} \\ 2 & 1 & \mathbf{-2} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{0} & \mathbf{0} & 1 & 4 \\ -1 & \mathbf{0} & \mathbf{0} & 3 \\ -4 & -3 & \mathbf{0} & \mathbf{0} \end{pmatrix}$ ………….. (14)

and the first matrix of right hand side of (14) is concrete symmetric and second one is concrete skew-symmetric. Clearly the expressions (13) and (14) are distinct.

Again consider another example in which

$A = \begin{pmatrix} 2 & 1 & -1 & -1 & 5 \\ -6 & 3 & 2 & 5 & 0 \\ 4 & -1 & 3 & 2 & 6 \end{pmatrix}$. Then $A = \begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{-1} & -\frac{7}{2} & \frac{9}{2} \\ -\frac{7}{2} & \mathbf{3} & \mathbf{2} & \mathbf{5} & -\frac{1}{2} \\ \frac{9}{2} & -\frac{1}{2} & \mathbf{3} & \mathbf{2} & \mathbf{6} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{5}{2} & \frac{1}{2} \\ -\frac{5}{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$ ……………. (15)

and the first matrix of right hand side of (15) is concrete symmetric and second one is concrete skew-symmetric.

Again $A = \begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{-5} & -2 & 5 \\ -2 & \mathbf{3} & \mathbf{2} & \mathbf{4} & 0 \\ 5 & 0 & \mathbf{3} & \mathbf{2} & \mathbf{6} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{0} & \mathbf{0} & 4 & 1 \\ -4 & \mathbf{0} & \mathbf{0} & 1 \\ -1 & -1 & \mathbf{0} & \mathbf{0} \end{pmatrix}$ ……………….. (16)

and the first matrix of right hand side of (16) is concrete symmetric and second one is concrete skew-symmetric. Clearly the expressions (15) and (16) are distinct.

3. CONCLUSION : Further study may be continued on the ring $(CM(F), \oplus, \odot)$ or $\left(M_\rho(F), +, .\right)$.

REFERENCES :

1. Biswas Jayanta : *Extension of Matrix Algebra and Linear spaces of Linear Transformations,* International Journal of Novelty Research in Physics, Chemistry and Mathematics, ISSN-2394-9651, Vol-3, Issue 1, pp : 29-42. (2016)
2. Connell E. H., *Elements of Abstract And Linear Algebra,* 2004.
3. Lay David C., *Linear Algebra And Its Applications,* (4th Edition). Pearson Education Inc, 2012.
4. Matthews K. R., *Elementary Linear Algebra,* Second Online Version, 1998.
5. Meyer Carl D., *Matrix Analysis and Applied Linear Algebra,* SIAM, 2004.

WEB REFERENCES :

- http://www.noveltyjournals.com/journal/IJNRPCM/Issue-1-January-2016-April-2016/0
- http://www.math.miami.edu/~ec/book/book.pdf
- https://app.box.com/s/fz732jl3lg0q4jk32bh0no0h84bkbssa
- http://www.zuj.edu.jo/download/elementary-linear-algebra-k-r-matthews-pdf/
- https://www.cse.msu.edu/~zhangh40/mth/Matrix_Analysis.pdf