The Mathematics of Epicycloid and Trochoid

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Abstract: *This paper focuses on the geometry of calculus, with the aim of providing a project for the students in calculus. Calculus plays a vital role in many engineering applications, such as engines, gears and mechanisms. There are numerous methods for analyzing mechanisms based on geometrical constructions. Our aim is to deepen the study of the curves described by a point and the relationship between the geometry of different parts. The study of curves describes the planets in motion. This paper has focused on drawing the mechanical curves epicycloids and trochoid, that are mostly used in engineering. Some engineering applications of these mechanical curves have been also studied. The characteristics of the parametric equations and numerical illustrations relating to trochoids and epicycloids are also discussed*

Key Words: *Calculus, Curves, Representation, Epicycloid, Trochoid.*

1. INTRODUCTION:

The study of curves originated from Ancient Greek, because the Greeks were the first who studied the paths that describe planets in motion. Galileo originated the term *cycloid* and he attempts first to make a sincere study of the curve. Nowadays cycloidal gears are commonly applied in clockwork and in pumps and root blowers. The tooth profile of the gears is composed usually of an epicycloid curve from the rolling circle up to the outside circle and a hypocycloid curve from the rolling circle to the root circle. Trochoid is defined as a curve generated by a point fixed to a circle, within or outside its circumference, as the circle rolls along a straight line. If the point lies outside the rolling circle, then the resultant curve is called an inferior trochoid and when the point lies outside the circle, it is called superior trochoid. The characteristics of trochoids prove very useful in determining the exact tooth fillet shape generated by racks consisting of circular arcs and straight line segments. In this paper, we have constructed the mathematical framework in which any further creative exploration could occur.

2. LITERATURE REVIEW:

This article provides the geometric relationships between subsequent cycloids and provides the corresponding properties of the curves. From David. M.Freeman [1], we understand the concept that as the continued fraction grows in length, the corresponding cycloids grow in diameter. M.Florez.et.al [2] gives a substantial knowledge about the concepts and standard equations of cycloids,epicycloids,hypocycloids. Frank Morely [3] helped me in obtaining a basic understanding of the methods and concepts in finding areas and surface areas under any trochoid arc. Hao Liu.et.al [4] explained that the outer rotors of the Gerator pumps are the circular arc trochoidal profile while their inner rotor profile is the conjugate curve of the outer rotor. Leon.M.Hall [5] assisted to acquire a detailed note on the concepts of speed,motion and acceleration about a trochoid. These artvciles motived us to study the concepts and problems of epicyloids and trochoids.

3. METHOD:

In this section, we devise a visualizing mathematical framework of epicycloids and trochoids.

- The derivations obtained on equations of epicycloid are widely used in many engineering applications .
- Construction of the length of the arc length is applied in finding miles traveled by airplanes,area of objects (Eg:Pizza).
- The concept of finding the minimum and maximum speeds of the point of a trochoid is used in analysing the speed of typical rotary pumps.
- The area under arc of a trochoid is used in finding the area traced by a windshield wiper blade.

4. NUMERICAL ILLUSTRATION:

Illustration 1:

Construct the equations for the epicycloid generated by a circle of radius b rolling on a circle of radius a

Explanation:

The parametric equations of the epicycloids is

$x = (a + b)\cos\emptyset - b\cos\left(\frac{a+b}{b}\emptyset\right)$

$y = (a + b)\sin\emptyset - b\sin\left(\frac{a+b}{b}\emptyset\right)$

A polar equation can be derived by computing

$x^2 = (a + b)^2\cos^2\emptyset - 2b(a + b)\cos\emptyset\cos\left(\frac{a+b}{b}\emptyset\right) + b^2\cos^2\left(\frac{a+b}{b}\emptyset\right)$

$y^2 = (a + b)^2\sin^2\emptyset - 2b(a + b)\sin\emptyset\sin\left(\frac{a+b}{b}\emptyset\right) + b^2\sin^2\left(\frac{a+b}{b}\emptyset\right)$

so

$r^2 = x^2 + y^2$

$= (a + b)^2 + b^2 - 2b(a + b)\{\cos[(\frac{a}{b} + 1)\emptyset]\cos\emptyset + \sin[(\frac{a}{b} + 1)\emptyset]\sin\emptyset\}.$

But

$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$,

So,

$r^2 = (a + b)^2 + b^2 - 2b(a + b)\cos[(a/b + 1)\emptyset - \emptyset]$

$= (a + b)^2 + b^2 - 2b(a + b)\cos(a/b\,\emptyset)$.

Note that ϕ is the parameter here, not the polar angle. The polar angle from the center is

$\tan\theta = \frac{y}{x}$

$$= \frac{(a + b)\sin\emptyset - b\sin(\frac{a+b}{b}\emptyset)}{(a + b)\cos\emptyset - b\cos(\frac{a+b}{b}\emptyset)}$$

To get n cups in the epicycloid, b = a/n, because then n rotations of b bring the point on the edge back to its starting position.

$r^2 = a^2[(1 + \frac{1}{n})^2 + (\frac{1}{n})^2 - (\frac{1}{n})(1 + \frac{1}{n})\cos(n\emptyset)]$

$= a^2[1 + \frac{2}{n} + \frac{1}{n^2} + \frac{1}{n^2} - 2(\frac{1}{n})(1 + \frac{1}{n})\cos(n\emptyset)]$

$= a^2[\frac{(n^2 + 2n + 2)}{n^2} - \frac{2(n+1)}{n^2}\cos(n\emptyset)]$

$= \frac{a2}{n^2}[(n^2 + 2n + 2) - 2(n+1)\cos(n\emptyset)]$,

So,

$$\tan\theta = \frac{a\left(\frac{n+1}{n}\right)\sin\emptyset - \frac{a}{n}\sin[(n+1)\emptyset]}{a\left(\frac{n+1}{n}\right)\cos\emptyset - \frac{a}{n}\cos[(n+1)\emptyset]}$$

$$\tan\theta = \frac{(n+1)\sin\emptyset - \sin[(n+1)\emptyset]}{(n+1)\cos\emptyset - \cos[(n+1)\emptyset]}$$

An epicycloid with one cup is called a cardioid, one with two cups is called a nephroid, and one with five cups is called a ranuculoid.

Illustration 2:

Let c be the epicyloid curve parametrized by $\alpha : [0,2\pi] \rightarrow R^2$ and defined by $\alpha(t) := (5\cos t - \cos 5t, 5\sin t - \sin 5t)$

Compute the arc length of C .

Explanation:

Consider the known results:

i) cos 4t = cos t cos 5t + sint t sin 5t
ii) $\sin^2 t = 1 – \cos 2t / 2$

Given x(t) = 5 cost – cos 5t ➔ x'(t) = -5sin t + 5sin 5t

Also given y(t) = 5 sint – sin 5t ➔ y'(t) = 5cos t – 5cos 5t

The integral for arc length is given by

$= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\ dt$
$= \int_0^{2\pi} \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}\ dt$
$= \int_0^{2\pi} \sqrt{25\sin^2 t - 50\sin t \sin 5t + 25\sin^2 5t + 25\cos^2 t - 50\cos t\cos 5t + 25\cos^2 5t}\ dt$
$= \int_0^{2\pi} \sqrt{50 - 50\cos 4t}\ dt$
$= \int_0^{2\pi} \sqrt{50(1 - \cos 4t)}\ dt$
$= \int_0^{2\pi} \sqrt{50\,(2\sin^2 2t)}\ dt$
$= 10*4 \int_0^{\pi/2} \sin 2t\ dt$

= 40

Illustration 3:

Compute the minimum and maximum speeds of the point of a trochoid and the locations where each occurs.

Explanation:

The parametric equation of a trochoid is given by
x(t) = at − bsint ; y(t) =a – bcost, where a is the radius and b is the distance from center of circle
For a parametric curve, the speed is just the modulus of the tangent vector.
Hence if
γ(t)=(at − bsint, a − bcost)
we have
γ'(t)=(a − bcost, bsint)
so
$v^2(t) = a^2 + b^2 - 2abcost$
and the stationary points for v(t) are the ones for which cost = ±1:
Hence Max v(t)=|a+b|
Min v(t)=|a−b|.

Illustration 4

The area under one arch of a trochoid x = 6θ – 3 sinθ, y = 6 – 3 cosθ is 81∏

Explanation:
Given a parametric curve x = f (t), y = g (t), a ≤ t ≤ b.
Area under the curve is obtained using the relation $A = \int_a^b g(t)f'(t)dt$

Here the given equation are x = 6θ – 3 sinθ, y = 6 – 3 cosθ

$$A = \int_0^{2\pi} (6 - 3\cos\theta)(6 - 3\cos\theta)d\theta$$

$$= \int_0^{2\pi} (36 - 36\cos\theta + 9\cos^2\theta)d\theta$$

$$= \left(36\theta - 36\sin\theta + \frac{9\theta}{2} + \frac{9\sin 2\theta}{4}\right)_0^{2\pi} = 81\,\prod$$

5. CONCLUSION:

In this article, the polar form of equations of the epicyloid and its arc length are derived. Also the minimum and maximum speed of the arc of trochoid is explained. The area of one arc of the trochoid is found by the application of the integral calculus.In future these concepts can be extended to anlayze the properties and relation between one arc of the curve and its generating circle.

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