The Study Of Cartography in Four Colour Theorem

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Abstract: *This paper deals with the study of map colouring that is four colour theorem. Along with a application of four colour theorem in cartography and small example to show a detailed explanation of four colour theorem by colouring Indian map*

Key Words: *graph, graph colouring, map, map colouring, the four colour theorem.*

1. INTRODUCTION:

The problem of graph colouring geographical political maps has historically been associated with the theory of graph colouring. In the middle of the 19th century the following question was posed: how many colours are needed to colour a map in a way that countries sharing a border are coloured differently. The solution has been reached by linking maps and graphs. It took more than a century to prove that 4 colours are sufficient to create a map in which neighbouring countries have different colours. It is about assigning a colour to graph elements: vertices, edges, regions, with certain restrictions. With this paper we would like to assess the elements of the theory of graph colouring with an emphasis on its application on practical problems in the field of cartography. A mathematical basis for map colouring will be given along with the chronology of proving **The Four Colour Theorem**. In addition, world political map will be shown, to determine the minimum number of colours needed to colour a map properly in practice.

2. Graph theory basic concepts and definitions:

Definition 2.1

A graph G consists of a finite non-empty set V = V(G) whose elements are called **vertices, points** or **nodes** of G and a finite set E = E(G). If pairs of distinct vertices called **edges** of G. Such a graph we denote G(V;E) when emphasizing the two parts of G, (Fig. 1).

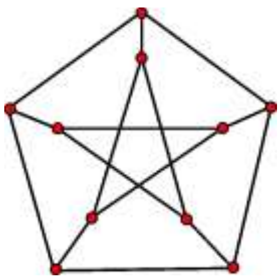


Figure 1: Example of a simple graph - The Petersen graph

Definition 2.2

An edge e = u,v is said to **join** the vertices u and v, and is usually abbreviated to e = uv. In such a case, u and v are called endpoints and they are said to be **adjacent**. Further, vertices u and v are said to be **incident** on e and vice versa, the edge e is said to be **incident** on each of its endpoints u and v.

Definition 2.3

A graph having parallel edges is known as a **Multiple graph.** A line joining the same points are called **multiple edges.** Graph having with no loops and no parallel edges is called a **simple graph.**

Definition 2.4

A bipartite graph 'G' then, G = (V, E) with partition V = {V1, V2} is said to be a **complete bipartite graph** if every vertex in V1 is connected to every vertex of V2.

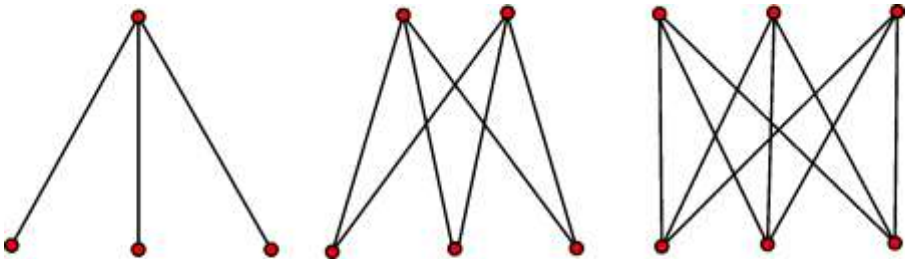


Figure 2: Complete bipartite graphs $k_{1,3}, k_{2,3}, k_{3,3}$

Definition 2.5

A graph G is **connected** if it cannot be represented as the union of two graphs. Otherwise, it is **disconnected**. Any disconnected graph can be represented as the union of connected graphs called **connected components** of G. A graph is said to be **finite** if it has a finite number of vertices and a finite number of edges, otherwise it is **infinite.**

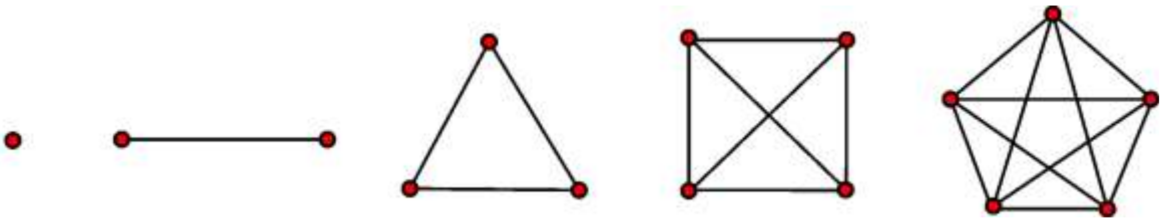
Definition 2.6

The **degree** of a vertex v in G, written deg (v), is equal to the number of edges in G incident with v. It shall be taken conventionally that a loop contributes 2 to the degree of v. A vertex of degree zero is called an **isolated vertex** and a vertex of degree 1 is an end-vertex.

Definition 2.7

A graph G is said to be complete if every vertex in G is adjacent to every other vertex in G. A complete graph with n vertices is denoted by K_n, (Fig. 3).

$K_n : s$ is used to denote a complete graph with |V|= n and |Ej|= s. It is easy to check that K_n has s = $\frac{n(n-1)}{2}$ edges.



K1 : 0 K2 : 1 K3 : 3 K4 : 6 K5 : 10

Figure 3: Some complete graphs

Definition 2.8

A **walk** in a graph G is an alternating sequence of vertices and edges of the form $v_0, e_1, v_1, e_2, \dots . e_k v_k$, where each edge e_i contains the vertices v_{i-1} and e_i, 1 ≤ i ≤k. In a simple graph a walk is determined by a sequence $v_0, v_1, \dots . v_k$, of vertices; v_0 being the initial vertex and v_k the final vertex. We say a walk is from v_0 to v_k, or connects v_0 to v_k. A walk is closed if the initial and final vertices are identified. The number k of edges in a walk is called its length.

Definition 2.9

A connected graph G is called **Eulerian** if there exists a closed trail containing every edge of G. Such a trail is called an **Eulerian trail**. A non-Eulerian graph G is **semi-Eulerian** if there exists a trail containing every edge of G.

Theorem 1 (Euler, 1736)

A connected graph G is Eulerian if and only if each vertex has even degree.

For the proof see e.g. [20].

Considering now a graph given in Figure 1 in the light of the above theorem, we conclude that the closed trail that meets the required conditions does not exist.

From the proof of Theorem 1 arises,

3. GRAPH COLOURING:

Definition 2.10

A(**Vertex) colouring** is an assignment of colors to the vertices of a graph 'G' such that no two adjacent vertices have the same colour. If no two vertices of an edge should be of the same color. It is a mapping c : V(G)→ S. The elements of S are called **colours**.

Definition 2.11

If a graph G is k-colourable, but not (k-1)- colourable, it is said that G is **k-chromatic**. The minimum number of colours needed to colour G is called the **chromatic number** of G and is denoted by $\chi(G),\chi(G)\leq|V|$

3.1 Planar graphs and maps

3.1.1 About planar graphs

Although graphs are usually presented two-dimensionally, i.e. in a plane, on paper or screen, it should be noted that each graph can always be presented in three-dimensional Euclidean space without the edges being crossed.

Definition 2.12

A graph is said to be **planar** if it can be drawn in a plane so that its edges do not cross, (Fig. 4).

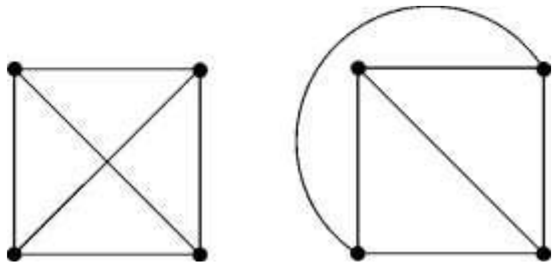


Figure 4: The complete graph k_4 is a planar graph and a map

Definition 2.13

A **map** is a connected planar graph where all vertices have degree at least 3. A map divides the plane into a number of **regions or faces** (one of them infinite). The term degree of a region, written deg(r), refers to the length of the cycle that surrounds it. Regions are said to be adjacent if they share an edge, not just a point.

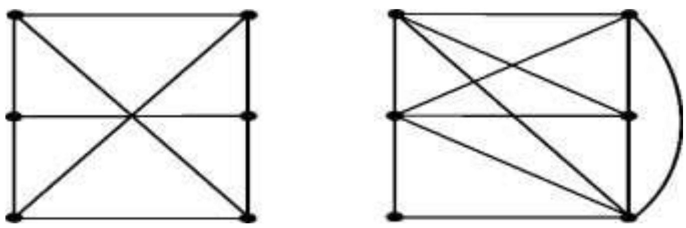
Definition 2.14

Graphs G and H are said to be **isomorphic** (G ≈ H) if there is a one-to-one correspondence between their vertices and their edges so that adjacent vertices are maped in adjacent ones. Two graphs are said to be **homeomorphic** if they are isomorphic or one from another can be obtained by removing or inserting vertices of degree 2.

Theorem 2 (Kazimierz Kuratowski, 1930)

A graph G is planar if and only if it contains no subgraph homeomorphic to k_5 or $k_{3,3}$

For example, graphs given in Fig. 5 are not planar.



(a) (b)

Figure 5: Examples of non-planar graphs

Indeed, graph (a) is graph $k_{3,3}$, (Fig. 6):

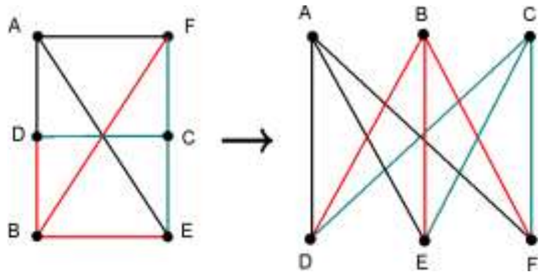


Figure 6: Graph $k_{3,3}$

while graph (b) is homeomorphic to graph K5, (Fig. 7):

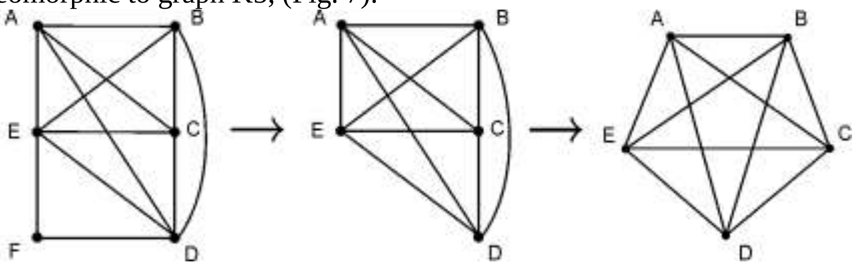


Figure 7: A graph homeomorphic to graph k_5

Theorem 3

k_5 is non-planar.

Proof

if possible, let k_5 be planar. k_5 contains a cycle of length five say (s,t,u,v,w,s)

Hence, without loss of generality, any plane embedding of k_5 can be assumed to contain this cycle drawn in the form of a regular pentagon. hence the edge (w,t) must lie either wholly inside the pentagon or wholly outside it.

Theorem 4

The graph k_5 and $k_{3,3}$ are not planar.

Proof

k_5 is a(5,10) graph

For any planar graph, q≤3p-6 . but e=10 and v=5 do not satisfy this inequality. Hence k_5 is not planar.

$k_{3,3}$ ia a bipartite graph and hence has no triangles. If such a graph is planar, then by q≤2p-4. But v=6 and e=9 do not satisfy this inequality. Hence $k_{3,3}$ is not planar.

3.2 Dual graph of a map

Definition 2.15

Map colouring is the act of assigning different colours to different regions (faces) of a map in a way that no two adjacent regions (regions with a boundary line in common) have the same colour. We now define a map to be **k-colourable** if its faces can be coloured with k colours. Similarity between the above definition and the one defining graph colouring is obvious. In order to show the colouring of a map to be equivalent to the vertex colouring we need a concept of the **dual map**, also known as
geometrical dual.

Theorem 5

Let K be a planar connected graph with v vertices, e bridges and r regions, and let its dual k^* has v^* vertices, e* bridges and r* regions. Then, v* = r, e = e*, r* = v.

Proof

V* = r follows at once from the definition of a dual graph. As there is a bijection between the edges of K and the edges of K*, we have e = e*, and as K* is planar and connected, applying Euler's formula one gets r* = 2-v*+e* = v *
As mentioned before, any colouring of the regions of a map
K correspond to vertex colouring of the dual K.As a concequence, we have the following result.

Theorem 6

A map K is region (face) k-colourable if and only if the planar graph of its geometrical dual K* is vertex k-colourable.

For the proof see [14].

Theorem 7

The four-colour theorem for maps is equivalent to the four-colour theorem for planar graphs.

For the proof see [20].

Theorem 8 (four color theorem)

Every planar graph is 4-colourable

A computer-free proof of the above theorem is still to be found

4. THE FOUR COLOUR THEOREM:

The four colour problem was defined as Francis Guthrie (1831-1899), the student of the University in London in 1852 was given the task to colour the map of English counties with as few colours as possible. He concluded that 4 colours were sufficient to complete the task with the counties sharing a common border being coloured with different colours in map. He wanted to find out whether each map in a plane or on a sphere can be coloured with 4 colours at the most with the neighbouring countries being coloured with various colours. It implies the fact that each country presents one coherent area. This question shall initiate a great number of attempts to find the answer by mathematicians and laypersons, which shall last for more than a century making this theorem one of the issues remaining unproven for the longest period of time. The main "tool" that the mathematicians will use in solving this problem will be the Graph Theory.

4.1 Historical overview

4.1.1 Francis Guthrie first noticed the problem

Although August Mobius, a German mathematician and astronomer mentioned the four colour problem in one of his lectures held in 1840, it is considered that Francis Guhrie first posed the problem. Francis Guthrie was a versatile person who was active in many areas. He was a very efficient barrister, acknowledged botanist (two plants were named after him: Guthriea capensis and Erica Guthriei), but first of all an excellent mathematician. However, since he could not find the solution to the four colour problem, he sent his notes with his brother Frederick to their mutual professor Augustus De Morgan. Augustus De Morgan (1806-1871) was a prominent English mathematician, a professor at the University in London who was very intrigued by this problem. Since
he did not know the answer, he wrote a letter on October 23, 1852 to his colleague and friend, Sir William R. Hamilton in Dublin where he presented the statement and gave an example showing that four colours suffice. He wrote in his letter as follows, (Fig. 8):

"A student of mine asked me to give him a reason for a fact which I did not know was fact - and do not yet. I cannot find an example where five colours are needed. If you retort with some very simple case, I think I must do as the Sphynx did.."

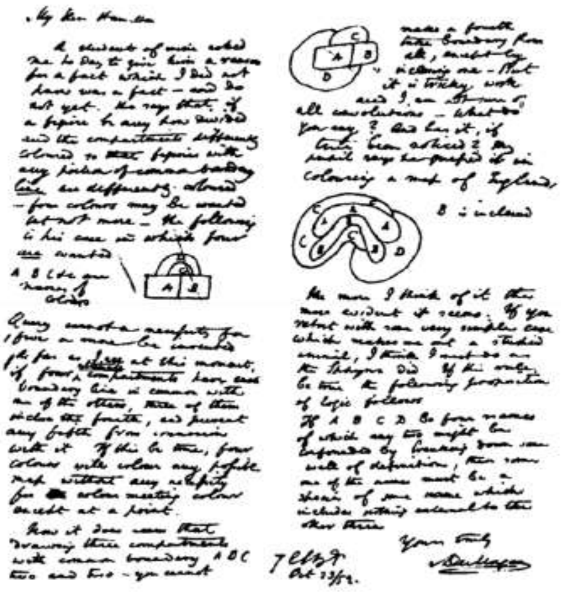


Figure 8: Display of the original letter

Sir Hamilton, however, was not interested. Therefore, De Morgan published the problem in 1860 in a literary journal Athenaeum. The American mathematician, philosopher and logician Charles Sanders Pierce learned about the problem probably from the journal, and tried to solve it. Although it was said that he managed to solve it, the proof has never been published.

5. Application within cartography:

Each map in the plane can be coloured with four colours such that neighbour areas are in different colours, as it is shown in previous chapters. Application of four colour theorem for colouring political maps is tested in this chapter. We use a world political map as an example. The software used is QGIS with its unofficial plugin TopoColour , which implements algorithms for graph colouring.

Geographical maps of administrative units, i.e. political maps, have some specialities that should be considered prior to the application of graph colouring algorithms. As it is defined in 3.1.1, neighbouring countries (administrative units) are those which shares common boundary line. Existence of common point does not imply neighbours. Geographical maps are abstract and generalized models of reality, and it is possible that some short boundary line in reality is represented as point, due to model or cartographic generalisation. Application is therefore possible only to model of geographical reality, users should be aware of these constraints, and in case of unexpected results, know how to deal with it.

Maritime boundaries are often not shown on political maps, and almost never are administrative areas on the sea coloured with different colours. This is certainly true for
world political maps, where colouring is usually applied to land parts of countries. This means that countries that share only maritime boundaries will not be considered as neighbours and could be coloured automatically with the same colour.

Further, countries are often consisted of more or less distant land parts, e.g. islands or exclaves. For example, some countries at certain administrative level contains overseas territories. This could potentially lead to non-planar graphs representing neighbours.

5.1 Methodology and programs used

The data used for the world countries were taken from GADM database of Global Administrative Areas. There is no unique solution for model of countries and its boundaries, and it is often result of the point of view of certain diplomacy.

For the purpose of this paper, we will not change the original data, because we will use it for the purpose of demonstration of application of graph colouring, and not for making special world political map. The dataset consists of 256 top-level administrative units, i.e. countries. Data was first loaded in QGIS as separate vector layer and transformed to Eckert VI map projection, one of map projections suitable for world maps. TopoColour plugin implements algorithms from graph colouring theory with purpose of colouring polygons in vector layer. It also allows creating graph representing detected neighbours in dataset.

5.2 Typical procedure for colouring areas in vector layer is as follows:

– Start the Topo Colour plugin and select polygon vector layer and one column in attribute table that has unique value for each administrative unit. Finding of neighbours starts. It can take a while for complex geometries (e.g. up to one hour or more).

– When neighbours are found, user selects “greedy” or “random” algorithm and starts the computing of colours. Number of colours needed is given as a result. For “random” algorithms, successive computations can give different number of colours.

– Save the colour numbers to one column in attribute table and style the layer.

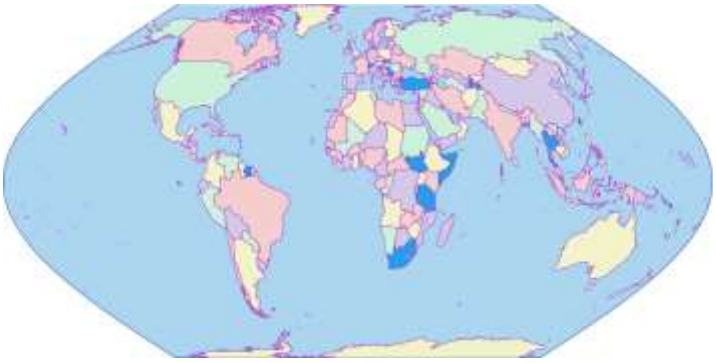


Figure 9: World political map coloured with five colours obtained by greedy algorithm implemented in QGIS plugin Topo- Colour. Less used colour is blue, and is a good candidate for manual elimination.

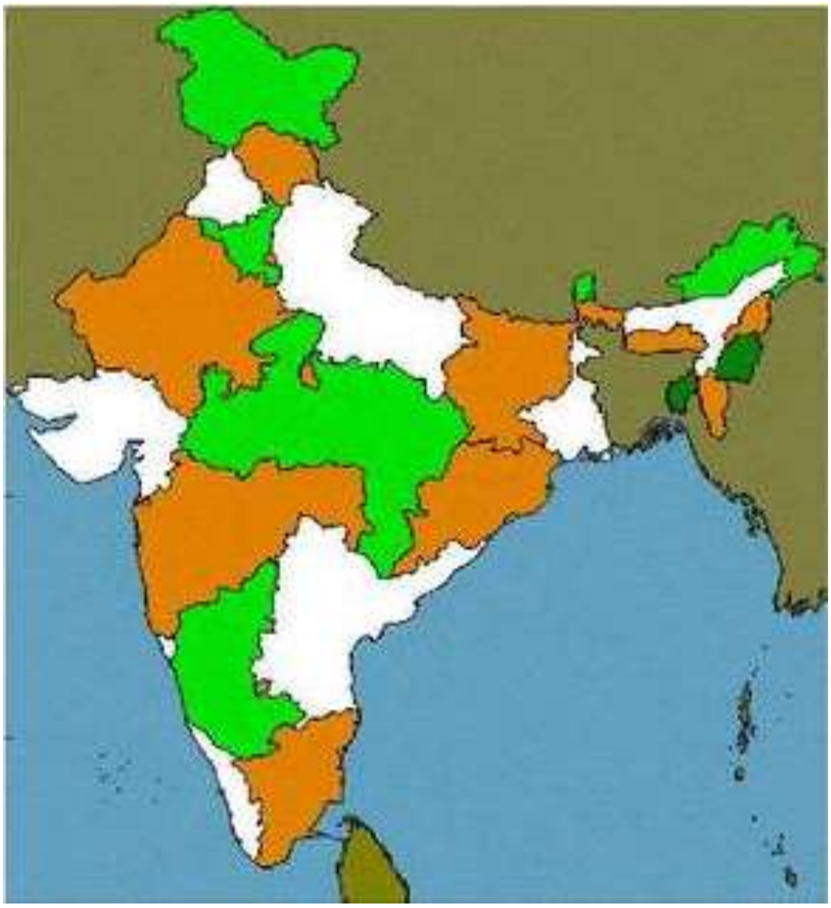


Figure 10: Manual elimination of the fifth colour from automatically coloured world political India map gives a map with four colour.

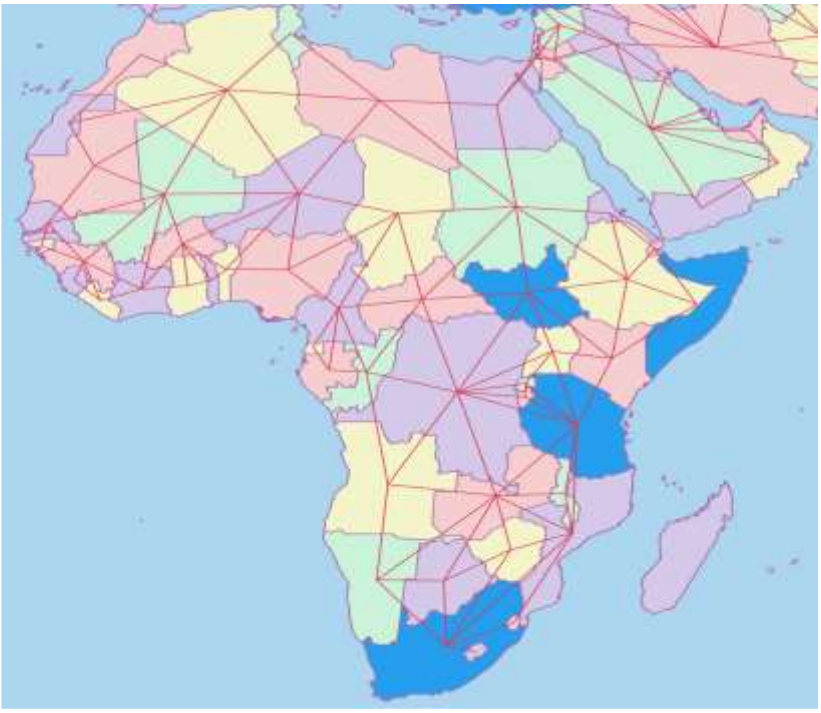


Figure 11: Graph representing neighbouring countries (clipped to Africa region)

6. CONCLUSION:

Graph colouring is widely applied in many scientific fields. In this paper, the focus is on the application in cartography. Since the map colouring with four colours is rarely mentioned in cartographic books, we were motivated to research this connection in this paper. The four colours conjecture has proved to be one of the greatest and long lasting problems in mathematics. The problem itself has attracted the attention both of mathematicians and laypersons. It took more than one century to prove this conjecture, which was achieved only with the development of information science and with computer assistance. It is also the first more significant theorem that has been proved in such a way. The theorem faced a lot of negative comments because of that and was not well accepted by the mathematical public of that time.

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