Received Date: 14/01/2020 Acceptance Date: 26/01/2020 Publication Date: 31/01/2020

Quadratic Equations in the Medieval Period of Indian Mathematics

Joyce Kurian
Assistant Professor, Department of Mathematics, Wilson College, Mumbai 400 007.
Email – kurianjoycxe@yahoo.com

Abstract: *The general perception among scholars is that Quadratic Equations have evolved through a long journey from Egypt to Babylon and the Greek Civilization before emerging in their current form in the Modern Era. The contribution of other Civilizations is generally ignored. In this paper the researchers intend to examine the developments in the field of the history of the Algebra from the Ancient Period to the Classical Period of Indian Mathematical history with special emphasis on Quadratic Equations.*

Keywords: *Quadratic Equations, Greek and other civilizations, Sulva Sutra, Bakshali Mathematics, Aryabhata*

1. INTRODUCTION:

In the past, many western scholars were of the opinion that Greek Mathematics had influenced the development of the field of Mathematics in India. A systematic attempt was made to study Greek Mathematics because of the availability of literature. But there was a lack of research in the area of non- European Mathematics till the 18th century, which the work of G. R. Kaye (1915), Colebrook (1813), Bibhutibhushan Dutta (1929), Hayashi (1997) and others led to interest in the history of Indian mathematics.

Kaye and others have questioned the originality and antiquity of the Indian contribution. Boyer's reading of Indian Mathematics can be understood from the statement, "Aryabhata had no feelings for logic or methodology and Brahmagupta treated irrational numbers, displaying innocence rather than mathematical insight" [2]. Al-Biruni's reaction of Indian Mathematics which has been echoed by later scholars reflects the fact that Indian Mathematics has a long history accompanied by puzzlement and frustration. Cajori [3] in his book ,'A History of Mathematics', says that Indians were more advanced in Arithmetic and Algebra than Greeks but in Geometry they were far behind their counterparts in Greece. In his book 'A History of Mathematics,' Katz [12] has given more importance to the history of Mathematics in other civilizations rather than Indian Mathematics. The reason for this may be that Indian Mathematics was considered as a by-product of Astrology or Astronomy. It was developed for rituals and calendars rather than as a discipline.

Egyptian Mathematics (Geometry and Algebra) was based more on practical applications like land redistribution due to floods caused by the rivers and construction of the pyramids. Greeks were particular about reasoning, proofs and demonstrations. They were very systematic and well versed in Geometry. Abstract Algebra was their weakness. Greek Mathematics or Science was accessible to the common people, anybody was free to study the discipline. Babylonians appeared to be very good in Arithmetic and Geometry. The availability of tablets written in Cunie form script shows that the people of the Tigro-Euphrates region had contributed immensely to the development of Arithmetic. Babylon was a great commercial center. The merchants, must have used Arithmetic for calculation to a very high level. Like Indians, they were also worshippers of Heavenly Bodies and great astronomers. But in the case of Indian Mathematics, it is very difficult to decide its antiquity due to unavailability of records as archaeological facts and manuscripts. Few manuscripts which are available for the later period bear testimony to the fact that Indian Mathematics reached its peak in the Classical Period. The Indian system of knowledge was not in the public sphere as in Greece but was in the hands of priests known as Saints living in Ashrams on the outskirts of settlements. Children belonging to the ordinary social strata as well as from royal backgrounds had to stay with the saints in the Ashrams to study Mathematics. The tradition of imparting knowledge was oral. A direct communication was set up between student and teacher. The results were in the form of verses, which were easy to memorize but were inaccessible to the common people.

The oldest Mathematics text available is the Alimes Manual, composed in 1700 BC, which includes linear and some quadratic equations. Pythagoreans were given the credit for knowledge in the form of ratio of sides to diagonals as $1:\sqrt{2}$ in the construction of squares. They did not accept the irrationals as the number system. Even with geometrical proof, the value of $\sqrt{2}$ was not accepted.

Later, in the period of Euclid Geometry (365 BC), it became necessary to obtain the approximate value of irrationals, hence the methods were developed. Geometry was their main field. Archimedes (287 BC) gave the value $\sqrt{3} \cong 0.7320513$ while finding the ratio of circumference of a circle to diameter. But his result was in the form of a result only, no suggestion or approach was given. After a long gap of a century, Heron (10 AD) used a formula for calculating square roots $\sqrt{a^2 + b}$ approximated to $a + \frac{b}{2a}$ as

$$\sqrt{50} \cong 7 + \frac{1}{14} \text{ ie } 7.071$$
$$\sqrt{63} \cong 8 - \frac{1}{16} \text{ ie } 7.937$$
$$\sqrt{75} \cong 8 + \frac{11}{16} \text{ ie } 8.687 \quad [6]$$

The last Greek method of finding the square root is found in the commentary written by Theon of Ptolamy Almagest in 125 AD. It finds the square root using sexagesimal system of angles given by the Babylonians. At the end of B.C, Greek started to treat Algebra as an abstract science but could not reach a consistent development till Diophantus introduced the symbol for operations for Unknown Quantity. Most of his work deals with the solution of indeterminate equations including quadratics. He accepted rational and irrational roots but refused to accept negative or imaginary values.

The ancient text available in Chinese Mathematics is in 'Nine Chapters on the Mathematical Art' (≅ 100 BC), authored by Jiu Shang Suanehu. This contains the proof of the Pythagoras Theorem and the formation of quadratic equations as,

$x^2 + (b\text{-}a)\, x = \frac{1}{2}[c^2 - (b-a)^2]$, from this the author has tried to find a general method for square roots. The ancient Chinese used geometrical models, using arithmetic of counting rods, to express their algebraic notions. [7]

As early as 800 BC Indian Mathematics dealt with square roots in the Sulba Sutras. The technical terms used in the Sulba Sutras are dvi-karami $\sqrt{2}$, tr-karami $\sqrt{3}$, astadasa-karami$\sqrt{18}$ etc. This period involved a lot of complex rituals and most of the scholars were priests, who themselves were very strict about the time period of rituals and construction of vedis for sacrifices. Vedis were required to be built with bricks. With the power of a mantra, a specific deity was invoked, once taking away the part of sacrifice offered. Particular deity departed from the site on his chariot, the shape was changed for other rituals without changing the area. Geometry was used to validate the algebraic results. The end of BC (between 200 BC to 100 BC) there was the rise of Jaina period, and the direction turned from square, rectangles and trapezium to circles. Indian Mathematics came to be known as Chord Geometry. Amma [1] has mentioned some formula involving chord *c, d* the diameter of the circle, *a* the arc, *h* the height of the segment as

$$c = \sqrt{4h\,(d-h)}$$
$$a = \sqrt{6h^2 + c^2}$$

and circumference of a circle $= \sqrt{10d^2} = (\pi d)$

(π was considered in Hindu Mathematics as $\sqrt{10} \cong 3$)

Then came the Classical Period (Christian Era) where the shift was towards algebraic analysis. Geometry was used to convince others. This period was dominated by the famous mathematicians Aryabhata, Brahmagupta, Sridharacharya, Mahavira, and Bhaskaracharya.

2. LITERATURE REVIEW:

Dutta, A. K. (2002): In the article, "Mathematics in Ancient India" the author is intend to have a glimpse of some of the landmarks in ancient Indian Mathematics with special emphasis on algebra. Here the author explained how algebra was emerged and developed.

Francis, C. (2007): In this research paper "Ancient Polynomial Equations", the writer Francis Cyriac has described about the development of polynomial equations. According to him the Egyptians and the Babylonians had contributed a lot to the progress of Mathematics.

John T. (2006): In the topic, "Mesopotamia: The Beginnings of Algebra", the author says that, Algebra is one of the oldest of all branches of Mathematics. The author mentioned in his book "Algebra-Sets, Symbols and The Language of Thought", how algebra was used about 4000 years ago in Mesopotamia. He said that Mesopotamians were a mathematically sophisticated people. They solved many important problems in ways that were quite different from the way we would solve with algebra. According to the author, the Mesopotamians moved from one problem to the next. Hence they advanced from the simple to the complex.

Joran, F. (2013): In the paper "Geometric division problems, quadratic equations, and recursive geometric algorithms in Mesopotamian Mathematics" the author shows how the need to find solutions of quadratic equations arouse in Mesopotamia. Here the author argued that, the tool used for the first exact solution of a quadratic equation will be either the use of the "Conjugate rule" or "completion of the square". In this paper, the author also discussed the recursive geometric algorithms for the solution of various problems related to division figure, which is the most significant achievement of some anonymous Babylonian mathematicians. In the entire paper, we can observe the author's discussion on the persistence of the Mesopotamian mathematical tradition over three full millennia. Here we observe that, the study of problems involving quadratic equations of different kinds and different degrees of complexity played a central role. The amazing rise of mathematical astronomy was the most blossoming part of Mesopotamian mathematics.

3. METHOD:

The methodology adopted in this research study is not one- dimensional. For this, the researcher will be utilizing historical, comparative, analytical and interpretive research procedure, for which primary and secondary sources will be used. Here the primary source will be speeches and the secondary sources used will be Reference Books, Research Books, Theses, Magazines, Journals, drawings, maps and photos. Books, Journals, Theses and Magazines available in University Libraries, Regional Libraries and World Wide Web will be the main source.

4. Ancient Indian Mathematics up to the Classical Period can be divided in to three Eras :

(1) Early Medieval Era (3000 BC – 500 AD)
(2) Classical Era (500 AD – 1200 AD)
(3) Modern Era (1200 AD – Till date)

4.1. Early Medieval Era

This period includes the Indus Civilization with the famous centers of culture at Harappan, Mohenjodaro, Lothal, Dholavira, Kalibangan, the Sulba Sutra period, the Jaina Mathematics as well as Bakshali Mathematics. The Indus Civilization did not show any written evidence of engagement with the field of Mathematics but its presence is seen in the construction of cities, construction of buildings followed standardized measurements of bricks in the ratio of 4:2:1. Difficulties in establishing connections arise because the script of the Harappans has not yet been deciphered. Sumerian records identify the trading of objects and material imports from certain regions showing the proximity of the two civilizations. Standard weights and length measurement scales were known to the Harappan Civilization in the decimal system[11]. Such an advanced civilization must have had a strong mathematical base and background in precision. A Babylonian tablet discovered (≅ 3000 BC to ≅2000 BC) showed the evidence of instruments, terminology related to astrology and knowledge of astronomy, which reached to its highest peak in ~ 700 BC surpassing to the Greeks. So it is difficult to say that the Greeks influenced the Indians. Like the Babylonians, Indian Mathematics was developed for Tithis and the calendar system. The effect of Indian Mathematics on the Babylonians cannot be ruled out even in the absence of evidence.

4.2 Sulba Sutra Period

The period after the Indus Civilization, was known as later the Vedic or Sulba Sutra period (800 - 200 BC). This period was dominated by priests as rituals became complex. Priests needed to construct altars and fire places prescribed for rituals. Geometry was the main criteria through which algebraic formulae were developed as Vedis were constructed in the shape of a square, rectangle, rhombus, kite or circular.

In Sulba Sutra, square roots of any number such as $\sqrt{2}, \sqrt{3}$ have been found in the construction of squares and rectangles. Apstama and Katyayana have formulated the rules which give the value of the diagonal if breadth and length are given. The result could be extended to the length of the diagonal $\sqrt{a^2 + b^2}$ if the rectangle is drawn with the sides *a* and *b* [1]. Apstamba used two bamboo rods equal to one purusa and $\sqrt{2}$ purusas respectively for constructing a square of area, 1 sq. purusa. Purusa is the unit of length used in Sulba Sutra. Sulba Sutra gives the value of $\sqrt{2}$ approximate to 1.414. The measure should be increased by one third of itself which again is increased by its one fourth and diminished by $\frac{1}{34}$ of that increment.

$$\sqrt{2} \cong 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}\ .$$

Thibaut's hypothesis of considering $2(12)^2 + 1 = 17^2$ depends on a square of 17 units cutting along two sides of $\frac{1}{34}$ units

$$(17 - \frac{1}{34})^2 = (17)^2 - 1 + (\frac{1}{34})^2$$
$$(17 - \frac{1}{34})^2 \cong (17)^2 - 1$$
$$\cong 288 = 2(144) = 2\,(12)^2$$

Taking square roots,

$$17 - \frac{1}{34} \cong \sqrt{2}\ .\ 12$$
$$12 + 4 + 1 - \frac{1}{34} \cong \sqrt{2}\ .\ 12$$
$$\sqrt{2} \cong 1 + \frac{4}{12} + \frac{1}{12} - \frac{1}{34.12}$$
$$\sqrt{2} \cong 1 + \frac{1}{3} + \frac{1}{4.3} - \frac{1}{4\ .\ 3\ .\ 34} \qquad [8]$$

$\cong$ 1.4142156863, which is correct up to 5 decimals.

Bhakshali Mathematics (~ 200 AD) also has the reference of finding square roots in the following algorithm.

In the case of a non- square number, subtract the nearest square number, divide the remainder by twice. Half the square of that is divided by the sum of the root and the fraction and subtract. [5]. The statement can be simplified as

$\sqrt{x} = x_0 + \frac{h}{2x_0} - \frac{h^2}{4(2x_0^2 + h)}$ where h = difference of nearest root square value ie x^2 - x_0^2

The result is true only for large numbers.

Example:-

Let us consider x^2 = 43205

x = 43205, x_0 = 207

h = x^2 - x_0^2 = 43205 - $(207)^2$ = 356

$= 207 + \frac{356}{2\,(207)} - \frac{(356)^2}{4\,(2(207)^2 + 356\,)}$

≅ 207.85990338 – 0.001778683

≅ 207.8581247, which is correct to 6 decimals.

4.3 Classical Period

The Classical Period from 400 AD to 1200 AD can be considered as a Golden Period in the history of Indian Mathematics, when it reached its peak. There was a long gap between the Sulba Sutra Period and the Classical Period. The ideas cultivated in this period left a great impact on Mathematical history. This period saw the rise of two contemporary mathematicians Aryabhata (476 AD) and Varahmihir (482 AD), who left a mark on mathematicians who came after them. Aryabhata was Acharya of Nalanda University and Varahmihir worked at a Gurukul established by his father Adityadasa. Varahmihir in his famous work Pancasiddhanta used algebra in a simple form. Aryabhata composed Aryabhatiya which consists of a full section of Ganitapada on Mathematics at the age of 23. Later, the work was carried on by Brahmagupta (Brahmasphuta Siddhanta 598 - 665 AD), Bhaskara I [600 - 680 AD, commentary of Aryabhata (Aryabhatiya Bhasya), Mahabhaskariya and Laghu bhaskariya], Lalla [720 - 790 AD, Shishyadhividhidantra, Jyotisaratnakota]

The 9th century saw five prominent Mathematicians Govindaswami, Mahavira, Prthudakasvami, Sankaranarayan and Sridhara. They were influenced by Aryabhata and Brahmagupta and carried their work forward. Mahavira (800-870 AD) had revised the Brahmasphuta Siddhanta in his work Ganitasangraha. Prthudakasvami (830-890 AD) is a prominent mathematician who has worked on solving the equations. Due to the Mughal invasions, the continuity of work in the North was shifted to the South where the Kerala School of Mathematics came into prominence. 9th to 11th century, most mathematicians, belonged to this school. Bhaskara II (1114-1185 AD) was the prominent mathematician at the end of the Classical Era whose impact can be seen in the Modern Period also. He was the Head of the Astronomical Observatory of Ujjain.

5. Quadratic Equations:

Aryabhata can be considered as the inventor of Algebra in Hindu mathematics. Aryabhatiya contains solutions of quadratic equation as well as algebraic relations, and indeterminate equations of first degree.

The following Sutra [10, Sutra 23] shows knowledge of expansion of $(a + b)^2$and $(a - b)^2$. Aryabhata must have

संपर्कस्य हि वर्गोधिशोधयेदेव वर्गसंपर्क।
यत्तस्य भवत्यर्धं विधादगुणकार सवंर्गम ।।

Here संपर्क (means addition), संपर्कवर्गयोग (addition of squares), सवंर्गम (product), राशि (quantity). The English translation of this Sutra is “From the square of the sum of two numbers subtract the sum of squares and then divide by 2. The result obtained is the product of two numbers”.

Parmeshwara, in his commentary, has given the following example:-

The sum of two numbers is 12. Its square is 144, sum of the squares 74, subtract this from 144 answer is 70, divide it by 2, we get 35 which is the product of 5 and 7. Generalizing this, we get,

$$ab = \frac{(a+b)^2 - (a^2 + b^2)}{2}$$

The Sutra [10, Sutra 25] of Aryabhatia

मूलफलं सफलं कालमूल गुणमर्धमूल कृतियुक्तम।
मूलं मूलार्धनं कालहनं स्यातस्वमूल फलम ।।

States the rule on how to calculate the interest amount on the principal. This is the solution of a quadratic equation. The Sanskrit word मूल (principal), मूलफल (extra amount paid over principal amount, that is, interest), सफलं (principal + interest, that is, Amount), काल (time for which amount is given), Amount = A, Time period = t, Principal = p

Sutra states that, multiply Amount, time and principal and then add half of principal square $Apt + \left(\frac{p}{2}\right)^2$. Take the square root and subtract half of the principal.

According to Parmeshwara,

Amount = 16, period = 6 months, product is 16 × 6 = 96. If principal is 100, then *Apt* = 9600. Add the square of half of 100. That is, 50^2. That is, 2500.

9600 + 2500 = 12100. A square root of this is 110. Subtract half of principal 50, result is 60. Divide 60 by 6 (time 6 months) = Interest for one month is 10.

In general,

$$\sqrt{Apt + \left(\frac{p}{2}\right)^2} - \frac{p}{2}$$

divide it by time *t*, result is interest I for one month. $I = \frac{\sqrt{Apt + \left(\frac{p}{2}\right)^2} - \frac{p}{2}}{t}$, which is the solution of the quadratic equation $tx^2 + px - Ap = 0$

Let us consider

$$x = \frac{\sqrt{Apt + \left(\frac{p}{2}\right)^2} - \frac{p}{2}}{t}$$

$$x\,t = \sqrt{Apt + \frac{p^2}{4}} - \frac{p}{2}$$

$$t\,x + \frac{p}{2} = \sqrt{Apt + \frac{p^2}{4}}$$

$$\left(tx + \frac{p}{2}\right)^2 = Apt + \frac{p^2}{4}$$

On simplification gives the quadratic equation,

$$tx^2 + px - Ap = 0$$

Aryabhata has used the quadratic equations, but he has not given solutions anywhere. Brahmagupta was more systematic in giving solutions. Brahmagupta has given rules as follows:-

वर्गोचतुर्गुणितां रूपाणं मध्यवर्गसहितानाम।
मूलं मध्येनोनं वर्गद्धिगुणोदवृतं मध्य।। [9, Sutra 44]

Multiply 4 to the absolute quantity and coefficient of the square of unknown quantity add square of middle coefficient and then decrease by middle coefficient, divide by twice of the coefficient of unknown quantity. If the quadratic equation is $ax^2 + bx = c$, then solution in the modern form is

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

The Sutra

उत्तरहीन द्धिगुणादी रोषवर्ग धनोतरा ष्टवधे।
प्रक्षिप्य पदं शेषोनं दृिगुणो त्तरहतं गच्छ।। [9, sutra 18]

Brahmagupta has given the rule to calculate the value of a number of terms of arithmetic progression.

English translation of the rule is:- From twice the initial term (*a*) subtract an increment (*d*), square the remainder (R) and add it to 8 times the product of the sum(*s*) and increment(*d*). Take the square root and from it subtract the remainder (R) and whatever the result divide it by 2 times increment (*d*).

Example: Citi (Bhata) has 10 bricks. Increment was of five bricks. Then the total became 100 bricks. Find n ?

Solution:- $a = 10$, $d = 5$, $s = 100$, $n = ?$

$R = 2a - d = 2(10) - 5 = 15$

$R^2 = (15)^2 = 225$

$8\,s\,d = 8 \times 100 \times 5 = 4000$

$8\,s\,d + R^2 = 4000 + 225 = 4225$

Square root = 65

$$n = \frac{\sqrt{8sd + R^2} - R}{2d} = \frac{65-15}{2d} = \frac{50}{10} = 5$$

In modern notation

$$n = \frac{\sqrt{8sd + (2a-d)^2} - (2a-d)}{2d}$$

$$2nd + (2a-d) = \sqrt{8sd + (2a-d)^2}$$

Squaring both the sides

$$4n^2d^2 + (2a-d)^2 + 4nd(2a-d) = 8sd + (2a-d)^2$$

or $4n^2d^2 + 8and - 4nd^2 = 8sd$

or $dn^2 + 2an - dn = 2s$, which simplifies to the quadratic equation,

$$dn^2 + (2a-d)n - 2s = 0.$$

Sridhara (750AD) indicated his method of solving quadratic equation. His work is not available but mentioned in Bhaskara and others [4]

Dutta has translated it as " Multiply both the sides (of an equation) by a known quantity equal to four times the coefficient of the square of the unknown, add both the sides the known quantity equal to the square of the (original) coefficient of the unknown; then extract the root".

Quadratic Equation is $ax^2 + bx = c$

Unknown is x and x^2 is the square of unknown having coefficient a

And coefficient of unknown is b.

$$4a(ax^2 + bx) = 4ac$$

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2$$

$$(2ax + b)^2 = 4ac + b^2$$

$$2ax + b = \sqrt{b^2 + 4ac}$$

$$x = \frac{-b + \sqrt{b^2 + 4ac}}{2a}$$

He did not consider the negative sign. Sripati (1039) and Bhaskara II had the same approach. Bhaskara II considered the two values. According to him the quadratic equation had two solutions. He considered $\pm$ sign with square root.

6. Different forms of equations:

In his Sutra, Aryabhata has considered the differences and product of two quantities to find them.

द्विकृति गुणासं संवर्गादि दृयन्तरवर्गोण सयुंतान्मूलम।
अन्तर युक्तंम हीनं तद्गुणकार दृथं दलितम।। [10, Sutra 24]

Commentator Parmeshvara has given the following example:-

Let us consider the product of two terms = 10, multiply 10 by 4 = 40 difference of terms is 3. Its square is 9. Addition of 40 and 9 = 49, square root is 7. To this 7 add the difference 3, result is 10.

दलितम means $\frac{1}{2}$. That is, $\frac{10}{2} = 5$. It is one quantity, the square root value is 7- 3 = 4; divide by 2. Second quantity is 2.

We put this in algebraic language.

If x and y are the quantities.

Then $x = \frac{\sqrt{4xy + (x-y)^2} + (x-y).}{2}$

$y = \frac{\sqrt{4xy + (x-y)^2} - (x-y)}{2}$

If we put $x - y = a$, $xy = b$, these values become roots of the equation, $x^2 - ax - b = 0$.

Brahmagupta has given the same result. According to Dutta, Narayan has given the same rule as, "The square root of the square of the differences of two quantities plus four times their product is their sum". [4]

7. CONCLUSION:

The Indian influence and contribution to the field of Mathematics has not been fully explored. Several practical difficulties have stood in the way of in depth studies on the subject. The fact that the work is available in cryptic Sanskrit, alphabets were used for numerals and also that the Sutras are compact in form and difficult to translate. Also words have different hidden meanings which cannot be explained without adding or subtracting something. But it is a fact that scholars insightful knowledge regarding mathematical calculations, and without this Astronomical studies would not have been possible. The researchers in this paper have considered very few cases in their study based only on Aryabhata's work. But much needs to be done to explore and establish that Indian Mathematics were not far behind their counterparts in other parts of the world.

REFERENCES:

1. Amma, S.,(1979). Geometry in Ancient and Medieval India 1st ed., Delhi: Motilal anarsidass Publishers.

2. Boyer, C.B., (1968). A History of Mathematics 1st ed., New York, USA: John Wiley& Sons.Inc.
3. Cajori, F., (1919). A History of Mathematics 2nd ed., New York: The Macmillan Company.
4. Datta, B. B., & Singh, A. N. (1938). History of Hindu Mathematics, A source book. Mumbai, Maharashtra: Asia Publishing House.
5. Dutta,B.B.(1929). The Bhakshali Mathematics, Calcutta, West Bengal: Bulletin of the Calcutta Mathematical Society.
6. Karl, F., 1900. A brief history of Mathematics 1st ed., Chicago: The Open Court Publishing Company.
7. Ling, W. & Needham, J., (1955). Horner's method in Chinese Mathematics:Its root in the root extraction procedures of the Han Dynasty, England: T'oung Pao.
8. Murthy, V.,(2005). Indian Mathematics in Sanskrit: Concepts and Achievements, Tirupati, Tamilnadu: Rashtriya Sanskrit Vidyapeetha.
9. Sharma, A. R. (Ed.). (1966). Brahma-sphuta Siddhauta with Vasana, Vijnana and Hindi Commentaries (Vol. I and III). Banglore, Karnataka: Indian Institute of Astronomical Research.
10. Singh, U. N. (1906). The Aryabhatiya. Bidhupur, Bihar: Madhuapur.
11. Thapar, R., (2000). Cultural pasts:Essays in early Indian History, New Delhi: Oxford University Press.
12. Victor Katz, (2007). The Mathematics of Egypt, Mesopotamia,China,India and Islam: A source book 1st ed., New Jersey, USA: Princeton University Press.