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Fixed Point of Continuous Mapping in Complete Metric Space

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Abstract: *Fixed point theory is a interesting subject, with an unlimited number of applications in different fields of mathematics. The purpose of this paper is to study for finding the fixed points of continuous mappings in complete metric space also we discuss the uniqueness and existence of fixed point for a family of continuous types mappings defined on complete metric space. We study some fixed point theorem for continuous type mappings in a complete metric space, it is shown that the same algorithm converges to a fixed point of a continuous type mappings under suitable hypotheses on the coefficients. Here the assumptions on the coefficients are different and techniques of the proof are also different.*

Key Words: *Complete metric space, contraction mapping, Equicontinuous, convex set, Lipschitz continuous.*

1. INTRODUCTION:

The problem of existence of fixed points of continuous type mappings in complete metric space is now a classical theme. The applications to fixed point of continuous type mapping made it more interesting. In this paper, we proved some fixed point theorems for continuous type mappings of a complete metric space. Fixed point theorems involves maps g of a set Y into itself that, under special conditions, contains a fixed point, i.e., a point u ∈ Y such that g(u) = u. The existence of fixed points has proper applications in different branches of topology and analysis.

1.1. PRELIMINARIES: In the sequel we shall make use of the following notations, definitions, lemmas and theorems.

1.2.. NOTATION: Let X be a given normed space, u ∈ X and r > 0 we write

$$B_X(u, r) = \{v \in X : kv - uk < r\}$$

$$\overline{B_X}(u, r) = \{ v \in X : kv - uk \le r\}$$

$$\partial BX(u, r) = \{v \in X : kv - uk = r\}$$

Whenever misunderstandings might occur, we write $\|u\|_X$ to stress that the norm is taken in X. For a subset A of X, we denote by $\overline{A}$, A^C, span (A) and co(A) the closure of A, the complement of A, the linear span of A, and the convex hull of A respectively.

2. DEFINITIONS:

1. A mapping T: D → Y is called Lipschitzian if and only if

$$\|Tx - Ty\| \le L\|x - y\|$$
$$\forall\ x,y \in D \text{ some } 0 \le L < 1$$

2. A mapping T: D → Y, is called non expansive if

$$\|Tx - Ty\| \le L\|x - y\|$$
$$\forall\ x,y \in D \text{ some } 0 \le L \le 1$$

3. A nonempty subset D of metric space Y is said to be Convex if it satisfies the following axioms

$$x, y \in D \Rightarrow t\,x+(1-t)y \in D \quad \forall\ x, y \in D \text{ and } \forall\ t \in [0,1]$$

4. A metric space Y is said to be complete if every Cauchy Sequence in Y converges to a point of Y.

5. Let be a mapping T: X → Y and dom(T) be a subspace of x then the mapping from dom(T) to Y is called operator.

6. Let us assume that Y be a metric space having a distance d. A self map g on space Y is known as Lipschitz continuous if ∃ a constant $\gamma \ge 0$ such that $d(g(u_1), g(u_2)) \le \gamma\, d(u_1, u_2)$, $\forall\ u_1, u_2 \in Y$.

The smallest γ for which the above inequality satisfies is called the Lipschitz constant of g. If $\gamma \le 1$ theng is called non-expansive and if $\gamma < 1$ then g is called a contraction.

7. Let us suppose that Y be a metric space and distance d be a distance of Y. A self mapping f on metric space Y is known as weak contraction if $\forall u_1 \ne u_2 \in Y$

$$d(g(u_1), g(u_2)) < d(u_1), u_2),$$

3. THEOREMS:

1. Let us assume that Y be a complete metric space and let g be a self map on Y. For some $n \geq 1$,if g^n is a contraction mapping. Then g has a unique fixed point
$\bar{u} \in$ Y.

2. Let us suppose that Y be a complete metric space, and let g be a self continuous map on Y. Assume ∃ a function $\emptyset: Y \to [0, \infty)$ such that $d(u, g(u)) \leq \emptyset(u) - \emptyset(g(u))$, ∀ u ∈ Y. Then g possesses a fixed point in Y. Furthermore, for any $u_0 \in$ Y the sequence $\{g^n(u_0)\}$ converges to a fixed point of g.

3. Let us assume that Y be a complete metric space, and let g be a self map on Y such that
$d(g(u_1),g(u_2)) \leq \gamma \max (d(u_1, u_2), d(u_1,g(u_1)), d(u_2, g(u_2)), d(u_1, g(u_2)), d(u_2, g(u_1))$
for every $u_1, u_2 \in$ Y and some $\gamma < 1$. Then we find $\bar{u} \in$ Y as a unique fixed point of g.

4. Let us suppose that Y be a complete metric space, and let g be a self map on to space Y. Assume that ∃ a lower semi continuous function $\emptyset: Y \to [0, \infty)$ such that

$$d(u, g(u)) \leq \emptyset(u) - \emptyset(g(u)), \quad \forall u \in Y$$

Then g possesses at least one fixed point in complete metric space Y.

Our main results are follows:-

4. MAIN THEOREMS:

4.1. THEOREM:1. Let us assume that g be a contraction mapping on a metric space Y which is complete . Then f possesses a unique fixed point $\bar{u}$ in space Y.

Proof: Let u_1, u_2 be two fixed point of g in Y. Since g is contraction mapping therefore ∃ a constant $\gamma < 1$ such that
$d(u_1, u_2) = d(g(u_1), g(u_2)) \leq \gamma\, d(u_1, u_2)$
$\Rightarrow d(u_1, u_2)(1-\gamma) \leq 0$
$\Rightarrow d(u_1, u_2) = 0 \quad \because d(u_1, u_2) \geq 0$ and $\gamma < 1$
$\Rightarrow u_1 = u_2$.

Choose now any $u_0 \in$ Y, and define the iterate sequence $u_{n+1} = f(u_n)$ ∀ n=1,2,3… . By induction on n, $d(u_{n+1}, u_n) \leq \gamma_n d(g(u_0), u_0)$.mAgain choose n ∈ N and m ≥ 1 such that,

$$d(u_{n+m}, u_n) \leq d(u_{n+m}, u_{n+m-1}) + \cdot\cdot\cdot + d(u_{n+1}, u_n)$$
$$\leq (\gamma^{m+n} + \cdot\cdot\cdot + \gamma^n)\, d(g(u_0), u_0).$$
$$\leq \frac{\gamma^n}{1-\gamma} d(g(u_0), u_0).$$

Hence $\{u_n\}$ is a Cauchy sequence, and admits a limit $\bar{u} \in$ Y, for Y is complete.
Since g is continuous, we have $g(\bar{u}) = \lim_{n\to\infty} g(u_n) = \lim_{n\to\infty} u_{n+1} = \bar{u}$.
Consequences of main theorems are as follows:-

4.2. THEOREM 2: Let Y be a complete metric space and Z be a topological space.
Let g: Y × Z → Y be a continuous function. Let us assume that g be a contraction on Y uniformly in Z, that is,

$$d(g(u_1, v), g(u_2, v)) \leq \gamma\, d(u_1, u_2),\ \forall u_1, u_2 \in Y,\ \forall v \in Z$$

for some $\gamma < 1$. Then, for every fixed v ∈ Z, the map u → g(u, v) has a unique fixed point ψ (v). Furthermore, the function v→ ψ (v) is continuous from Z to Y.
We observe that if g: Y × Z → Y is continuous on Z and is a contraction on Y uniformly in Z, then g is continuous on Y × Z .

Proof: Keeping the fact in mind of main theorem, we only have to prove the continuity of ψ. For $v, v_0 \in$ Z, we have

$$d(\psi(v), \psi(v_0)) = d(g(\psi(v), v), g(\psi(v_0), v_0))$$
$$\leq d(g(\psi(v), v), g(\psi(v_0), v)) + d(g(\psi(v_0), v), g(\psi(v_0), v_0))$$
$$\leq \gamma\, d(\psi(v), \psi(v_0)) + d(g(\psi(v_0), y), g(\psi(v_0), v_0))$$

Which implies

$$d(\psi(v), \psi(v_0)) \leq \frac{1}{1-\gamma} d(g(\psi(v_0), v), g(\psi(v_0), v_0)).$$

Since the above right-hand side tends to zero as $v \to v_o$, we have the required continuity.

THEOREM:3. Let us suppose that Y be a complete metric space, and g be a self map of space Y. Assume that there exist a right-continuous function $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(r) < r$ if $r > 0$, and $d(g(u_1), g(u_2)) \leq \psi(d(u_1, u_2))$, ∀ $u_1, u_2 \in$ Y. Then g has a unique fixed point $\bar{u} \in$ Y. Furthermore, for any $u_0 \in$ Y the sequence $\{g^n(u_0)\}$ converges to $\bar{u}$. Clearly this result is special case of main theorem, for $\psi(r) = \gamma r$.

Proof: If $u_1, u_2 \in Y$ are fixed points of f, then

$$d(u_1, u_2) = d(g(u_1), g(u_2)) \leq \psi(d(u_1, u_2))$$

so $u_1 = u_2$. For proving the existence, fix any $u_0 \in Y$, and define the sequence $u_{n+1} = g(u_n)\ \forall\ n=1,2,3\ldots.$ We show that sequence $\{u_n\}$ is a Cauchy sequence, and the desired result comes by arguing like in the proof of Theorem 3.3.3 for $n \geq 1$, define the positive sequence

$$b_n = d(u_n, u_{n-1}) \quad \forall\ n=1,2,3\ldots.$$

It is clear that $b_{n+1} \leq \psi(b_n) \leq b_n$; therefore b_n converges monotonically to some $b \geq 0$. Right-continuity of ψ, gives $b \leq \phi(b)$ and this implies $b = 0$. If $\{u_n\}$ is not a Cauchy sequence, there is $\varepsilon > 0$ and integers $m_k > n_k \geq k$ for every $k \geq 1$ such that

$$d_k := d(u_{m_k}, u_{n_k}) \geq \varepsilon,\ \forall\ k \geq 1.$$

In addition, upon choosing the smallest possible m_k, we may assume that

$$d(u_{m_{k-1}}, u_{n_k}) < \varepsilon$$

For k big enough (here we use the fact that $a_n \rightarrow 0$). Therefore, for k big enough,

$$\varepsilon \leq d_k \leq d(u_{m_k}, u_{m_{k-1}}) + d(u_{m_{k-1}}, u_{n_k}) < b_{m_k} + \varepsilon$$

Which implies that $d_k \rightarrow \varepsilon$ from above as $k \rightarrow \infty$.
Furthermore,

$$d_k \leq d_{k+1} + a_{m_{k+1}} + b_{n_{k+1}} \leq \psi(d_k) + + b_{m_{k+1}} + b_{n_{k+1}}$$

and by taking the limit as $k \rightarrow \infty$ we get the relation $\varepsilon \leq \psi(\varepsilon)$, which is false since $\varepsilon > 0$.

THEOREM:4. Assume that each g_n has at least a fixed point $u_n = g_n(u_n)$. Let g be a uniformly continuous self map on Y such that g_m is a contraction for some $m \geq 1$. If g_n converges uniformly to g, then u_n converges to $\overline{u} = g(\overline{u})$.

Proof: We first assume that g is a contraction mapping (i.e., m = 1). Let $\gamma < 1$ be the Lipschitz constant of g. for given $\varepsilon > 0$, we can choose $n_0 = n_0(\varepsilon)$ such that $d(g_n(u), g(u)) \leq \varepsilon(1 - \gamma)$, for all $n \geq n_0$, and$u \in Y$.
Then, for $n \geq n0$,

$$\begin{aligned} d(u_n, \overline{u}) &= d(g_n(u_n), g(\overline{u})) \\ &\leq d(g_n(u_n), g(u_n)) + d(g(u_n), g(\overline{u})) \\ &\leq \varepsilon(1-\gamma) + \gamma\, d(u_n, \overline{u}). \end{aligned}$$

Therefore $d(u_n, \overline{u}) \leq \varepsilon$, which proves the convergence of sequence. To show the general case it is sufficient to see that if $d(g^m(u), g^m(v)) \leq \gamma^n d(u, v)$ for some $\gamma < 1$, now we define a new metric d_0 on Y which is equivalent to d given by

$$d_0(u, v) = \sum_{k=0}^{m-1} \frac{1}{\gamma^k} d(g^{k+1}(u), g^{k+1}(v))$$

Furthermore, since g is uniformly continuous therefore g_n also converges uniformly to g with respect to d_0. In the end, g is a contraction mapping with respect to d_0. Virtually,

$$\begin{aligned} d_0((g(u), g(v)) &= \gamma \sum_{k=0}^{m-1} \frac{1}{\gamma^k} d(g^k(u), g^k(v)) + \frac{1}{\gamma^{m-1}} d(g^{m-1}(u), g^{m-1}(v)) \\ &\leq \gamma \sum_{k=0}^{m-1} \frac{1}{\gamma^k} d(g^k(u), g^k(v)) = \gamma d_0(u, v). \end{aligned}$$

Now the problem is diminished to the last case for m = 1. The next result dispatch to a particular class of complete metric spaces.

THEOREMS:5. Let Y be locally compact and let as asseume that $\forall\ n \in N$ there is $m_n \geq 1$ such that $g_n^{m_n}$ is a contraction mapping. Let us assume that g be a self map on Y such that g_m is a contraction for some $m \geq 1$. If g_n converges point wise to g, and g_n is an equicontinuous family of functions, then $u_n = g_n(u_n)$ converges uniformly to $\overline{u} = g(\overline{u})$.

Proof: Let $\varepsilon > 0$ be arbitrary however small such that

$$L(\overline{u}, \varepsilon) = \{u \in Y : d(u, \overline{u}) \leq \varepsilon\} \subset Y$$

is compact. By the application of the Ascoli theorem,$\{g_n\}$converges to g uniformly on $L(\overline{x}, \varepsilon)$, since it is equicontinuous and pointwise convergent therefore for given $\varepsilon > 0$ choose $n_0 = n_0(\varepsilon)$ such that $d(g_n^m(u), g^m(u)) \leq \varepsilon(1-\gamma)$ for all n greater than or equal to n_0 and for all u in $L(\overline{x}, \varepsilon)$ where $\gamma < 1$ is the Lipschitz constant of g^m. Then for all n greater than or equal to n_0 and for all u in $L(\overline{u}, \varepsilon)$ we have

$$\begin{aligned} d(g_n^m(u), \overline{u}) &= d(g_n^m(u), g(\overline{u})) \\ &\leq d(g_n^m(u), g^m(u)) + d(g^m(u)), g^m(\overline{u})) \\ &\leq \varepsilon(1-\gamma) + \gamma\, d(u, \overline{u}) \\ &\leq \varepsilon \end{aligned}$$

Hence $g_n^m(L(\overline{u}, \varepsilon)) \subset L(\overline{u}, \varepsilon)$ for all n greater than or equal to n_0. Since the maps $g_n^{m_n}$ are contractions, it implies that, for $n \geq n_0$, the g_n has a fixed points u_n and belong to $L(\overline{u}, \varepsilon)$ i.e., $d(u_n, \overline{u})) \leq \varepsilon$.

5. CONCLUSION:

Finding fixed points of nonlinear continuous type mappings (especially, non-expansive mappings) has received vast investigations due to its extensive applications in partial differential equations, nonlinear differential equations. In this paper, we devote to construct the methods to finding the fixed points of continuous type mappings in the complete metric space. In most cases, we observed that fixed points suddenly appeared when they are required. On the contrary, we think that they should deserve a proper place in any general text book, especially in a functional analysis text book. This is the main reason that encourages me to write down this article. I tried to collect most of the meaningful results of the field from different papers. In this article, we have proved some fixed point theorems for the expansive mapping in the settings of complete metric spaces.

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