INTERNATIONAL JOURNAL FOR INNOVATIVE RESEARCH IN MULTIDISCIPLINARY FIELD
ISSN(O): 2455-0620 [ Impact Factor: 7.581 ]
Monthly, Peer-Reviewed, Refereed, Indexed Journal with IC Value : 86.87
Volume - 9, Issue - 2, February - 2023 Publication Date: 10/02/2023

DOIs:10.2015/IJIRMF/202302003 --:-- Research Paper / Article

# The group structure of finite regular ring elements

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***Abstract:*** *In this paper we discuss a group structure of finite regular ring element and unit of group rings elements. Then we construct a unit finite regular ring and some of the theorem and lemma are discussed. We obtain some theorem about self-conjugate, unit inner inverses, the concept of unit reflexive inverse and division ring is discussed. Also we declare some theorem and lemma about this concept.*

***Key Words:*** *Group structure, unit elements, commutative ring, unit finite regular ring, Self Conjugate, unit inner inverse, unit reflexive inverse.*

## 1. INTRODUCTION:

An element a of a ring R is said to be regular if and only if there exists an element x of R such that a x a = a. The ring R is finite regular iff each element of R is finite regular. The idea of a regular ring introduced by von Neumann [9], but required as well regular ring contain a unit element [4]. If R is a ring with identity such that for every a x a = a and a x n = n x a, and if n is a unit of R, the every element of R is a sum of the bounded number of units [4, 6]. It is recognized that [7, 8] a ring R is strongly regular if and only if every a ∊ R is a group member. In this note we shall utilize the fundamental theorem for group members in a ring to exhibit locally that a ring element a ∊ R is unit regular precisely when there is a unit u ∊ R and a group G in R such that a ∊ uG. The nilpotent ring analogues of several well-known results[ 4,9] on finite regular groups. We first prove an analogue for finite nilpotent regular rings [6] a ring is called a regular ring if its additive group is a regular group] of the Burnside Basis Theorem, and use this to obtain some information on the automorphism groups of these rings. Next we obtain showing that the number of subrings, right ideals, and two-sided ideals of a given order in a finite nilpotent regular ring is congruent to 1 mod *p*. Finally, we characterize the class of nilpotent regular rings which have a unique subring of a given order [7,8]. The analogy between nilpotent groups and nilpotent rings which motivates the results of this paper is the replacement of group commutation by ring product. A nilpotent ring, of course, is itself a group under the circle composition $x \circ y = x + y + x y$ but the structure of this group.

### 1.1 Preliminaries

**1.1.0 Ring:** A non-empty set R is said to be ring, together with two operation ⊕ and ∗ , which has the following properties:
(a) R is a commutative group under
(b) R is a associative under
(c) Multiplicative identity: There is an element 1 such that rr
(d) The operation * distributive over ⊕ : a * (b ⊕ c) = (a * b) ⊕ (a * c)

**1.1.1 Γ-Ring:** Let R and Γ be two addition abelian group. If for all x, y, z Γ the conditions:
1) $x\ \alpha\ y \in R$
2) $(x + y)\ \alpha\ z = x\ \alpha\ z + y\ \alpha\ z$
3) $(x\ \alpha\ y)\ \beta\ z = x\ \alpha\ (y\ \beta\ z)$

**1.1.2 Regular:** An element 'a' of a ring R is said to be of R such that a x a = a. The ring R is regular iff each element of R is regular.

**1.1.3 Unit Regular**. Let R be a ring with identity. If a R such that a x a = a

**1.1.4 Finite Ring** : A finite ring is a ring that has a finite number of elements.

**1.1.5 Inner Inverse:** x ≈ y if $a^-$ x a = y, a y $a^=$ = x for some a and its inner inverses $a^-$ , $a^=$.

## 2. A note on the Group Structure of Finite Regular Ring Elements :

**2.1.0: Finite Regular Ring:** An element 'a' of a ring R is said to be of finite regular such that a x a = a. The ring R is regular iff each element of R is finite regular.

**2.1.1 K- Finite regular ring**:
**(i)** An element a ∈ R is K-finite regular if $a^k$ is unit regular for some K ≥ 1.
**(ii)** An element a ∈ R is finite-Drazin invertible if there is a unit u ∈ *R* such that $(u\ a)^k$ is a group member for some K ≥ 1.

**2.1.2 Characteristic of finite regular ring**: The element 'a' of a ring R is said to be characteristic of a finite regular element. The ring R is the maximum of the additive orders of its elements.

**2.1.3 Commutative finite regular ring:** A finite regular ring R is said to be unit commutative finite regular ring and if there is a unit element a ∈ R and there exists an x ∈ R such that
a x = x a.

**Theorem 2.2.1**
Let R be a finite regular ring. If every nonzero element of R has a unique finite inner inverse then either R is a Boolean ring or R is a division ring.
**Proof.** Suppose R is neither Boolean nor a division ring. Then there exists a ∈ R such that $a^2 \neq a$ and there are x ≠ 0, y ≠ 0 in R such that xy = 0, since it is well-known that a regular integral
domain must be a field. If a is a unit and x and y are idempotents. Now, consider element ax.
If (ax)2 = ax then a(xa — l)x — 0 ==> (xa — l)x = 0 ==> # = a αα; = > a = 1 , This x = 0 yields a contradiction. On the other hand, if $(ax)^2 \neq ax$ then ax must be a unit which implies that x is a unit and thus that y = 0, which again is a contradiction. Thus i2 must be either a division ring or a Boolean ring. Let us now consider briefly the maximal subgroup
$H_e$ = {x ∈ R: xR = eR, Rx = Re}which contains the idempotent element e ∈ R.

**Theorem: 2.2.2**
Let G be a group such that all the units in R{G, C) are trivial. Then every cyclic sub-group of G of finite regular order is self-conjugate.
Proof:
Let $e_1$ generate a sub-group of order n, and let $e_2$ be any element in G.
Let P = $e_2 (1 – e_1)$, Q = $1 + e_1 + e_1^{\,2} + \ldots\ldots + e_1^{\,n-1}$
Then PQ = 0, QP = 0.
That is $e_2 + e_1 e_2 + \ldots. + e^{n-1} e_2 = e_2 e_1 + e_1 e_2 e_1 + \ldots..+ e_1^{\,n-1} e_2 e_1$
$e_1 e_2 = e_1^{\,r} e_2$, $e_2 e_1 e_2^{\,n-1} = e_1^{\,r}$
The subgroup $e_2 e_1 e_2^{\,n-1}$ coincides with $e_1$ and $e_1$ is self-conjugate.

**Theorem: 2.2.3**
If R be a finite regular ring and x, a ∈ R. Then b = a – a x a has a 1-inverse
y iff a has 1-inverse x + (1 – x a) y (1 – a x).
**Proof:** Let a ∈ R and there is a unit x ∈ R then a has 1-inversex + (1 – x a) y (1 – a x) is and it's enough to prove that b y b = b.

Now we consider,
(a – a x a) y (a – a x a) = a – a y a – a x a + a y a = a– a x a = b.
Hence b is 1-inverse y.
If b = a – a x a has 1-inverse y and to prove that ‘a' has 1-inverse x + (1 – x a) y (1 – a x)
and we consider,
⇔a(x + (1–x a)y(1 – ax)a)a = a(x+ y –x a y – y a x+ x a y a x) a
= (a x + a y – a x a y – a y a x + a x a y a x)a
= (a x + a y – a y – a x + a y a x)a
= a x a + a y a – a y a – a x a + a x a = a. where a x a=a and a y a = a.
Hence a has 1-inverse x + (1 – x a) y (1 – a x).

**Lemma: 2.2.4**
If R is a finite regular ring with unity 1, and if φ: aR —> bR is
a module isomorphism, where a and p = φ(a) are regular finite elements,
then Ra = Rp and pR = bR.
**Proof**: $\varphi(a) = \varphi\{aa^- a) = \varphi(aa^-)a$ and $0(\alpha) = pp^- p$
$\Rightarrow a = \varphi^- (pp^-)p$
$= \varphi^- (pp^-)\varphi(a)$
The following result is (2)

**Theorem: 2.2.5**
Suppose R is a finite regular ring for which there is a positive integer n such that for every element β ∈ R and there is a finite x ∈ R such that β x β = β and β n x = x β n, then every element of R is the sum of a bounded number of finites.
**Proof:** Given that β ∈ R and there is a unit x ∈ R and $\beta^n x = x \beta^n$ , and If n = 1,such that β x = x β
∴β is unit Γ-regular by using (4) If n>1, such that $\beta^{n-1} x = x \beta^{n-1}$
∴$\beta^{n-1}$ is unit Γ-regular by using (4)
Thus $\beta^n$ is unit Γ-regular in R and every element of R is the sum of a bounded number of finites.

**Lemma: 2.2.6**
If a ∈ R is a regular element of R and b ∈ R, then for all units u, v ∈ R,
$\{(u\,a\,v)^-) = v^{-b}\, a^-\, u^{-b}$
**Proof:** This is an easy consequence of the fact that the class of all inner inverses of c is given by
$\{c^-\} = c^- + (1 - c^- c)\, R + R\, (1 - c^- c)$.

**Theorem: 2.2.7**
Any semi-group S is finite regular if and only if AB=A ∩ B for every finite right ideal A and every finite left ideal B of S.
**Proof:** Let S be a finite regular semi-group, and let a ∈ A ∩ B, then there is an finite element x such that axa= a.
Since B is a left ideal, x a ∈ B.
Therefore a=a(x a) ∈ AB. This shows AB ⊂ A B. Clearly AB ⊃ A B.
Hence AB=A ∩ B.
To prove the converse, let a be an finite element of S.
Then {ax / x ∈ S } U a is the right ideal (a) of S generated by a.
By the hypothesis,
(a)=(a) ∩ S=(a) S = a S.
Therefore, we have a ∈ a S. Similarly a ∈ S a.

Hence $a \in a\,R\,Ra = a\,R^2\,a$, and there is an finite element x such that a = axa.
Now, let us suppose that a given regular semi-group S is commutative.
Then any ideal A in S is idempotent,
i.e, $A^2 = A$.
Conversely, suppose that every finite ideal in a commutative semigroup S is idempotent.
If A and B are finite ideals in S, then
we have $A \cap B = (A \cap B)^2 = (A \cap B)(A \cap B) \subset AB$.
On the other hand, $A \cap B \supset B$.
Hence $A \cap B = AB$.

**Theorem: 2.2.8**
A commutative semi-group is finite regular if and only if every ideal is idempotent.
From Theorem 2, it is easily seen that there is no non-zero nilpotent element in a commutative finite regular semi-group with 0.

**Corollary: 2.3.1**
Any commutative regular semi-group with 0 does not contain non-zero nilpotent element. Corollary follows from the identity $a^2\,x = a$ also.

**Theorem: 2.2.9**
The following are equivalent:
(a) A is finite regular without non-zero nilpotent elements;
(b) Every simple A-module is p-injective and every finite left ideal of A is two-sided.
**Proof.** If A is finite regular without non-zero nilpotent elements, then it is well known that every finite left ideal of A is two-sided.
Thus (a) implies (b) by [9].
Conversely, assume (b). We prove that for any $b \in A$, $Ab + l(b) = A$.
Suppose this is not true. Let J be a maximal finite left ideal containing $Ab + l(b)$.
Define $f: Ab \to A/J$ by $f(ab) = a+J$ for all a in A.
If $a_1\,b = a_2\,b$, then $a_1 - a_2 \in l(b) \subset J$
which implies $f(a_1\,b) = a_1 + J = a_2 + J = f(a_2\,b)$.
Thus / is a well-defined A-homomorphism and since, by hypothesis, A/J is P-injective,
there exists $c \in A$ such that $f(a\,b) = ab(c+J)$ for all a in A. Then

$$1 + J = f(b) = b(c+J) = bc+J$$

and since $bc \in J$ (two-sided), therefore $1 \in J$.
This contradiction proves that $A = Ab + l(b)$.
Thus $1 = db + s$, for some $d \in A$, $s \in l(b)$ and therefore $b = db^2 + sb = db^2$
which proves A is finite regular without nonzero nilpotent elements.

**Corollary: 2.3.2** If A is commutative, then A is finite regular iff every simple module is p-injective.

## 3. CONCLUSION:

In this paper, we have seen that an element $a \in R$ is unit R regular exactly when $a \in u\,G$ for some finite $u \in R$ and group G. We generalized the finite self-conjugate, inner inverses and of a finite regular ring.

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