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On an extended Petersen graph with a condition at distance two

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Abstract: *An L(2; 1) graph marking G is an integer labelling of the vertices in V (G) such that labels di2er by at least two for neighbouring vertices and labels di2er by at least one for vertices spaced apart by two. The minimal span over all L(2; 1)-labelings of G is what is known as the ß-number of G. In this study, we look at extended Petersen graphs' ß-numbers. We demonstrate that the ß-number of every generalised Petersen graph is limited from above by 9 by introducing the idea of a matched sum of graphs. Then, we demonstrate that, with the exception of the Petersen graph itself, this limit can be improved to 8 for all generalised Petersen graphs with vertex order 12.*

Key Words: *Generalized Petersen graph; 3-regular graphs; L(2; 1)-labeling; ß-number.*

1. INTRODUCTION:

A straightforward graph is G. An L(2; 1)-labeling of G is a mapping of L from V (G) into numbers such that| $L(v_2)$ -$L(v_1)$|< 2 if v1 and v2 are neighbouring in G and| $L(v_2)$- $L(v_1)$ < 1 if v1 and v2 are spaced two distances apart in G. Labels are the constituent parts of the image of L, and the span of L is the difference between the biggest and smallest labels. The ß-number of G is the minimal span over all L(2; 1)-labelings of G and is represented by the symbol ß(G).Additionally, if L is a labelling with a short spread, it is referred to as a ß-labeling of G. Without sacrificing generality, we will presume that the minimum label of G's L(2; 1)-labelings is 0.The issue of labelling a graph with a condition at distance two was initially researched by Griggs[9] and Yeh [11] as a variant of Hale's channel assignment problem [10]. In addition to obtaining bounds on the ß-numbers of graphs in classes like trees and n-cubes, they also took into account the connection between ß(G) and invariants (G), (G), and (G). They proved that ß(G)6 (G)+2 (G) and hypothesised that ß(G)6 (G) for ß(G)< 2. Chang and Kuo increased the Griggs and Yeh limit to 2(G) + in [3]. (G). Georges and others[5] linked the path-covering number of G^c to ß(G), proving that ß(G)6 |V (G)|-1 if and only if G^c has a Hamilton route. (Chang and Kuo [3] independently obtained this result as well.) Here, we describe a matched sum of graphs and determine bounds for its ß-number.[2]The generalised Petersen graphs, a particular class of matched sums, are the subject of our application of these findings, and we demonstrate that these 3-regular graphs have ß-numbers that are at most 9, supporting the [9]Griggs and Yeh hypothesis.

2. RESULTS :

Definition 2.1.1: Assume that V (G_1) and V (G_2) are graphs, and that M is a matching between them (G_2). The graph with V (G_1 M+G_2)=V (G_1)U V (G_2) and E(G_1 M+G_2)=E(G_1)U E(G_2) U M is then the M-matched sum (or simply the M-sum) of G_1 and G_2, written as G_1 M +G_2.We observe that if M =ϕ, then G_1 M + G_2 = G_1 + G_2 and that G_1 M + G_2 is a subgraph of G_1 U G_2(the union of G_1 and G_2).

Theorem 2.2.1: Assume that G1 and G2 are networks. For all M matches, the formula becomes
ß(G_1 M +G_2)6ß(G_1) + ß(G_2) + 2.

Proof: A labelling of G_1 M + G_2 with the span ß(G_1)+ ß(G_2) + 2 is sufficient to generate a L(2; 1)-labeling. Assume that G_1 is labelled as L_1, and G_2 is labelled as L_2. The following is how we define a marking L of G_1 M +G_2 : v ∈ V (G), then L (v);In the event that v ∈ V (G_2), L(v) = L_2(v) + ß(G_1) + 2:When i= 1; 2, the limit of L to V (G_i) is a L(2; 1)-labeling, and when i = 2; 3, the label of any vertex in V (G_1) di2ers from the label of any vertex in V (G_2) by at least 2.We can strengthen the upper limit adding a condition on the ß-numbers of G_1 and G_2 in relation to (G_1) and (G_2). This mild condition results from the following finding mentioned in [9]: ß(G) + 2 if G has a vertex of degree adjacent to two other vertices of degree. According to this conclusion, all k-regular graphs for k = 2 have ß-numbers that are at least k + 2.

Theorem 2.2.2: Assume that G1 and G2 are graphs with the property that ß(G_i)<(G_i)+2 for some i. Then, for all M matches, ß(G_1 M + G_2)6ß(G_1) + ß(G_2) + 1.

Proof: We presume that ß(G_1)<(G_1) + 2 without losing generality. Producing a L(2; 1)-labeling of G_1 M + G_2 with range ß(G_1) + ß(G_2) + 1 is sufficient. Let L_i be a ß-labeling of G_i for i = 1 and i = 2. We take into account the labelling L, which need not be a L(2; 1)-labeling, as defined as follows:L (v) if v >V (G); if v >V (G_2), L(v)= L_2(v) + ß(G_1) + 1We are done if L is a L(2; 1)-labeling.In the absence of such a non-empty subgroup, M, L fails to be a L(2; 1)-labeling because M′ = {g_1; g_2}, L(g_1) = ß(G_1) and L(g_2) = ß(G_1)+ 1. Let N (g_1) be the set of labels given by L to the neighbours of g_1 in G_1 for each {g_1; g_2}∈ M ′, and let W (g_1) = {0; 1; 2;:::; ß(G_1)- 2} N (g_1). Given that ß(G_1)<Δ(G_1)+2, ǀW (g_1)ǀ implies that W (g_1) is not vacant because dG(g_1)<(G_1)+2 _1 _Δ(G_1)=1.Therefore, by relabeling each g_2 with a label from W, we can create a new labelling L′. (g_1). Any two relabeled vertices have identical labels under both L and L_2, meaning that they are at least three distances apart. Since L′ has a spread of ß(G_1) + ß(G_2) + 1, it is clear that it is a L(2; 1)-labeling.

Corollary 2.3.1.

If G is a k-regular graph for k 1, then the following corollary holds: ß(GM + G)62ß(G) + 1 for all matchings M.

Proof: ß(G) =2 if k = 1, then. Inferring that GM + G is a graph with a maximal degree of at most 2, ß(GM + G)6 4 2ß(G)+1. If k = 2, then the conclusion is inferred from Theorem 2.2.1 by ß(G) = k + 2 from the debate that comes right before it.

Definition 2.1.2:

For n = 3, a 3-regular graph G of order 2n is referred to as a generalised Petersen graph of order n if and only if G is made up of two disjoint n-cycles, referred to as the inner and outer cycles, where each vertex on the outer cycle is adjacent to a (necessarily unique) vertex on the inner cycle. The terms "spoke" and "inner cycle" refer to the inner and outer cycles, respectively, and "edge joining an outer cycle vertex to an inner cycle vertex" refers to the edge connecting two vertex on the outer cycle. We refer to the group of n-order generalised Petersen graphs as GPG (n).

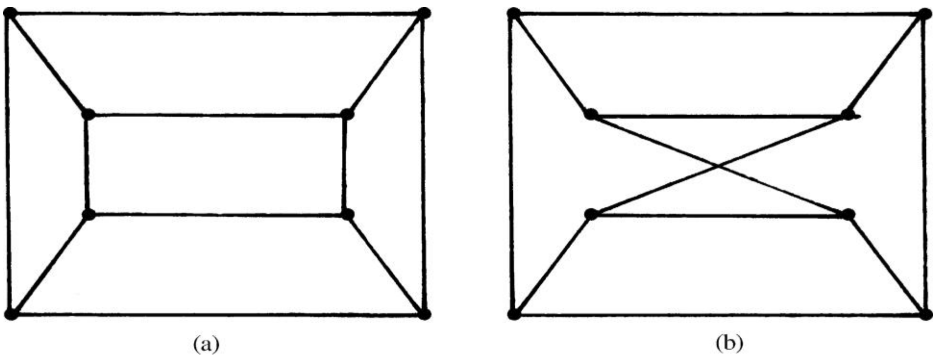


Fig. 1. The two graphs in GPG(4).

The two graphs in GPG(4) are illustrated in Fig. 1.

The Petersen graph PG is obviously a constituent of GPG(5), and each component of GPG(n) is an M-sum of two n-cycles, where M is a perfect matching. Therefore, from Carollary 2.3.1 it follows that ß(G)69 for all G in GPG because ß(Cn)=4 for n3 (n). In the scenario where =3, we draw attention to the fact that this upper limit is consistent with the Griggs and Yeh 2 conjecture.The extended Petersen graph K_2 Cn serves as the definition of the n-prism Pr(n) for n = 3. Figure 1a presents a 4-prism, and in [7], the following is proved.

Theorem 2.2.3:

For n= 3, 5 if n = 3k for some k; otherwise, ß(Pr(n)) = 6:We will acquire a more accurate upper bound on the ß-number of generalised Petersen graphs in the sections that follow.

Theorem 2.2.4:

We have ß(G)68 for k<1 and G ∈ GPG(3k).

Proof. Using the repeating pattern of 1, 3, and 5, we first create a labelling L (not strictly a L(2; 1)-labeling) by labelling the vertices on the outer cycle in ascending subscript order. $L(w_i)=2j + 1$ where j, i mod 3 (i.e. Similar to this, we label the inner cycle's vertices in ascending subscript sequence in accordance with the pattern of 0, 8, 6. We are done if L is a L(2; 1)-labeling.A maximal non-empty collection X of vertices with label 0, each of which is adjacent to a vertex with label 1, exists in the absence of L. Alternatively, a maximal non-empty collection Y of vertices with label 6, each of which is adjacent to a vertex with label 5, exists in the absence of L. By altering the labels of each vertex in X and Y from 0 to 4 and 6 to 2, respectively, we can create a L(2; 1)-labeling.

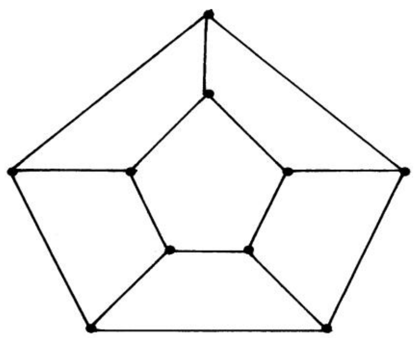
Theorem 2.2.5:
In the case of k=1 and G=GPG(3k + 1), ß(G)68.
Proof. By first creating a labelling L (not necessarily a L(2; 1)-labeling) such that $L(w_0) = 7$, we modify the approach in the proof of Lemma 2.7. The remaining 3k vertices on the outer cycle are then given labels in the repeating pattern of 1, 3, and 5 in ascending order of the vertex subscripts. Let v_0 be close to w_0 without losing meaning, and let $L(v_0)$ equal 2. In ascending order of the vertex subscripts, we label the unlabeled vertices on the inner cycle, observing that v_3 k is the only vertex labelled 6 under L within two distances of v_0 .Either a vertex with identifier 5 is nearby the vertex v_3 k, or it is not. If so, permuting the labels 1, 3, and 5 on the outer cycle will allow us to create a labelling L′ in which v_3 k is not close to a point with the label "5".In order to create a L(2; 1)-labeling of G from L′, the following steps must be taken: each vertex labelled 6 that is next to a vertex labelled 5 under L′ shall be relabeled as 2, and each vertex labelled 0 that is next to a vertex labelled 1 under L′ shall be relabeled as 4.

Lemma 2.4.1: ß(G)68 for k<2 and G ∈ GPG(3k + 2).
Proof. By first establishing a labelling L (not necessarily a L(2; 1)-labeling) such that $L(w_0)=L(w_4) =7$ and then assigning labels to the remaining 3k vertices on the outer cycle in the repeating pattern of 1, 3, and 5 in ascending order of the vertex subscripts, we modify the approach in the proof of theorem2.2.5 Without losing breadth, we allow v_0 to be near w_0 and w_4 to be near v_j for some j.

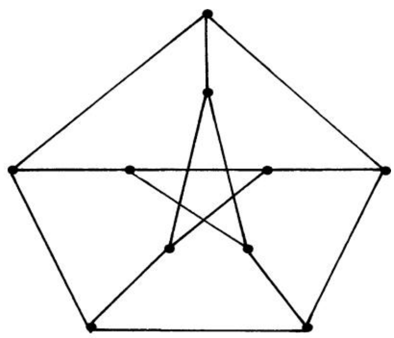
Suppose $L(v_0) = 2$ and $L(v_j) = 4$. We use elements of 0, 8, and 6 to label the remaining 3k inner cycle vertices in such a manner that exactly one vertex on the inner cycle labelled 6 is within two distances of vertex v_0 and exactly one vertex on the inner cycle labelled 0 is within two distances of vertex vj. Start by labelling the unlabeled vertices on the inner circle in ascending order of the subscripts of the as of yet unlabeled vertices, following the repeating pattern 0, 8, 6.

Then, precisely one vertex on the inner cycle with the label "6" is close to v_0 by a distance of two.Otherwise, exactly two vertices on the inner cycle labelled 0 exist within two distances of v_j, and as a result, precisely one vertex on the inner cycle labelled 8 exists within two distances of v_j. We achieve the intended labelling L by changing the repeating pattern from 0, 8, 6 to 8, 0, 6.

A vertex on the outer cycle with label 5 is either next to the singular vertex v with label 6 within two distances of v_0, or it is not. Similar to this, a vertex on the exterior cycle with label 1 is either next to a unique vertex v with label 0 within distance two of v_j, or it is not.We can create a labelling L′ where v is not next to a vertex with a label of 5 and v is not next to a vertex with a label of 1 by permuting the labels 1, 3, and 5 on the outer circle.Now, we create a L(2; 1)-labeling of G from L′ as follows: each vertex labelled 6 that is next to a vertex labelled 5 under L′ shall be relabeled as 2, and each vertex labelled 0 that is next to a vertex labelled 1 under L′ shall be relabeled as 4.



(a)



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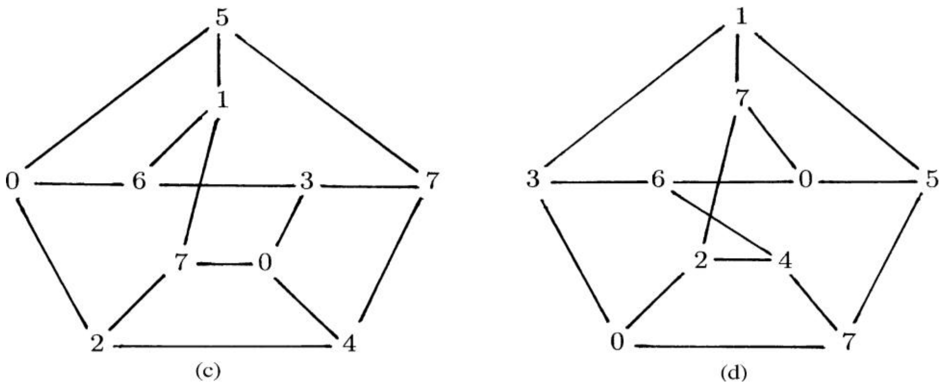

Fig. 2. The four graphs in GPG(5), along with labelings (c,d).

We can verify that GPG(5) =4 by noting that for n< 4, the diameter of each graph in GPG(n) is at least 4 (see also [1]). These four graphs are shown in Figure 2, along with L(2; 1)-labelings of the graphs in Figures 2c and 2d with a range of seven.The 5-prism in Fig. 2a has ß-number 6 Carollary 2.3.1. , making it a graph. The graph in Fig. 2b is the Petersen graph PG, and it has ß-number 9 because, given that PG has a diameter of 2, each of its 10 vertices has a unique label that is allocated by a L(2; 1)-labeling. Therefore, ß(PG)<9.

Proposition 2.5.1: Allow G ∈ GPG (n). If G is isomorphic to PG, then ß(G)=9. Otherwise, ß(G)68.
We observe that PG does not stand alone among generalised Petersen graphs only in the case of Lemma 2.4.1. As demonstrated by Castagna and Prins [2], all generalized According to their description, all Petersen graphs besides PG have a chromatic index of 3 (whereas ′(PG) = 4).We have not discovered a generalised Petersen graph with ß-number 8, setting an upper limit of 8 on the ß-number of generalised Petersen graphs other than PG. We observe that only Pr(3), which has ß-number 5theorem 2.2.3, is present in GPG(3).

Additionally, GPG(4) includes Pr(4) and the graph in Fig. 1b, the former of which has ß-number 6 (again by Theorem 2.2.3), and the latter of which has ß-number 7 by a different reasoning from the one that ß(PG) =9. In Figs. 2a, c, and d, the three GPG(5) PG members are depicted. It has already been established that they have ß-numbers 67. We then demonstrate that each component of GPG(6) has a ß-number of no more than seven.. As a result, we can assign the numbers 0, 4, and 6 to the inner cycle's vertices in ascending subscript order in accordance with some recurring pattern, such that $L(v_i)=L(v_i+3)=4$. We are done if L is a L(2; 1)-labeling. A maximum non-empty collection X of vertices with label 1 exists, each of which is next to a vertex with label 0, or a maximal non-empty collection Y of vertices with label 6, each of which is next to a vertex with label 7.By altering the labels of each vertex in X from 1 to 5 and each vertex in Y from 6 to 2, we can create a L(2; 1)-labeling.We can infer the following from the debate above.

Theorem 2.2.6:
When G ∈ GPG(n) with ß(G)=8, n is equal to 7.
Finally, we hypothesise that there is neither a generalised Petersen graph with ß-number 8 nor a 3-regular graph with ß-number 8. Additionally, according to our research, the Petersen graph is the only linked 3-regular graph with ß-number 9.

CONCLUSION:

In this paper discussed with demonstrate that the ß-number of every generalised Petersen graph is limited from above by 9 by introducing the idea of a matched sum of graphs. Then, we demonstrate that, with the exception of the Petersen graph itself, this limit can be improved to 8 for all generalised Petersen graphs with vertex order 12.

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