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A Study on Nonlinear Partial Differential Equations using Some Familiar Technique

Md. Harun-or-Rashid
Lecturer, Department of Electrical and Electronic Engineering, Khwaja Yunus Ali University, Sirajgonj-6751, Bangladesh
Email - harun.eee@kyau.edu.bd

Abstract: *The Nonlinear partial differential equations which may be ordinary or partial mainly describe the natural system all the complicated problems in the universe are almost generated by involving different types of differential equations with some boundary conditions or nonlinearities throughout the paper. We have analyzed and survey some nonlinear problems in connections with some familiar methods and equations such as "Euler-Lagrange's equation, charpit's technique. Obtaining some solutions of the nonlinear partial differential equations, we have observed that the relationship between the variations and how the initial or boundary conditions affect the results.*

Key Words: *Nonlinear Partial Differential Equations, Euler-Lagrange's Equation, Transport Equation, Charpit's Technique, First Integral Method.*

1. INTRODUCTION:

Nonlinear partial differential equations describe a huge portion of the physical systems that actually exist in the world (*A. Aftalion et al., 2006)*. Equations of this type appear in many different scientific disciplines, including but not limited to fluid mechanics, gas dynamics, combustion theory, general relativity, elasticity, thermodynamics, biology, ecology, neuroscience, and many more *(A.D. Polyanin et al., 2004).*

In this paper, we investigate a broad category of nonlinear differential equations and their numerical, complete, singular, and general solutions (L. F. Shampine,1994). Specifically, we do a discretization of the spatial variable in the corresponding partial differential equations *(Phoolan Prasad et al., 1985).* For nonlinear partial differential equations, a semi-discrete system is created, and we investigate the stability of the numerical method (P. Dr´ abek, 1992) and the convergence of the numerical solutions (L. F. Shampine, 1994). Nonlinear differential equations as a whole are analyzed here (A. Iserles, 1996). We illustrate our findings by presenting some instances of general class partial differential equations (R. J. LeVeque, 2007). We also used specific methods, like the Charpits method *(E. Hairer et al., 1996)* and the first integral method, to solve the nonlinear partial differential equation (E. Goursat, 2008).To be more specific, we extended our research to nonlinear partial differential equations and provided suitable examples to demonstrate our method *(A. Fasano et al., 2009).*

2. FUNDAMENTAL RESULTS:

We review some fundamental definitions and findings on Nonlinear Partial Differential Equations in this part.

2.1. NONLINEAR PARTIAL DIFFERENTIAL EQUATION:

An equation of the form $f(x , y , z , p , q) = 0$ is said to be a nonlinear partial differential equation. Where, The derivatives p and q separately having products.
For example,

1) $p^2 + q^2 = 1$, 2) $pq = 5$, 3) $p^2x + q^2y = z$

All are the non-linear partial differential equation.

Complete and singular solution of nonlinear partial differential equation: Let the non-linear partial differential equation of

$$f(x, y, z, p, q) = 0 \;\ldots\ldots\ldots.. (1)$$

Be derived from (1),

$$g(x, y, z, a, b) = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The removal of the two constants a and b. Consequently, (2) is referred to as the final answer to (1).
This full solution characterizes a family of surfaces with two parameters, one or both of which may define an envelope. Taking the opposite approach, we first remove a and b from
$g = 0$, $\frac{\partial g}{\partial a} = 0$, $\frac{\partial g}{\partial b} = 0$. If the eliminate λ (x , y , z) = 0 satisfies (1) , it is called the singular solution of 1); if $\lambda(x, y, z) = \xi(x, y, z) . \eta(x, y, z)$, And if $\xi = 0$ satisfies (1) while $\eta = 0$ is the singular solution *(P. Antonelli et al., 2012).*
Two-order non-linear partial differential equation: One approach to solve a specific second-order non-linear partial differential equation

$$F(x, y, z, p, q, r, s, t) = 0 \qquad \ldots\ldots\ldots\ldots (3)$$

The first step in the example was to find a relationship of the form

$$u = \psi(v) \qquad \ldots\ldots. (4)$$

Where, ψ is arbitrary constant $u = u(x, y, z, p, q)$ and $v = v(x, y, z, p, q)$
From which the specified differential equation might be constructed by removing the arbitrary function as an intervening variable. A relation of this type (4) is referred to as an intermediate integral of (3).

2.2 EULER – LAGRANGE EQUATION:

Let's imagine a mechanical system on R^n that has both kinetic $K(x, y)$ and potential energy U represented by the variables(x & y). The Lagrangian, $L(x, v)$: $R^n \times R^n$ →R, is defined as the minus the potential energy, U from the kinetic energy, K of the system. Trajectories of this mechanical system are said to minimize the action functional, according to the variational formulation of classical mechanics *(J. Guckenheimer et al., 2009).*

$$S[x] = \int_0^T L\big(x(t), x(t)\big)dt$$

More precisely, a C^1 trajectory x: $[0, T] \to R^n$ is a minimize S under fixed boundary conditions if for any C^1 trajectory y: $[0, T] \to R^n$ such that $x(0) = y(0)$ and $x(t) = y(T)$ we have

$$S[x] \leq S[y]$$

2.2.1 THEOREM:

Let, $L(x, v) : R^n \times R^n \to R$ be a C^2 function. Suppose that $x : [0, T] \to R^n$ is a C^2critical point of the action S under fixed boundary conditions $x(0)$ and $x(T)$. Then

$$\frac{d}{dt} D_v L(x, \dot{x}) - D_x L(x, \dot{x}) = 0$$

PROOF:

Let x be what it is said to be in the statement. After that, the function will work for any : [0, T] → R^n that has compact support on (0, T), the function, i(ϵ) = S[x + $\epsilon\varphi$], has a minimum at ϵ =0. Thus $i'(0)$ =0
That is $\int_0^T D_x L(x, \dot{x})\varphi + D_v L(x, \dot{x})\dot{\varphi} = 0$; Integrating by parts, we conclude that

$$\int_0^T \left[\frac{d}{dt} D_v L(x, \dot{x}) - D_x L(x, \dot{x})\right] \varphi = 0$$

For all φ : [0,T] → R^n with compact support in (0, T) *(E. Hairer et al., 1993).*
Hence the proof is complete.

2.3 NONLINEAR TRANSPORT EQUATION:

In order to solve nonlinear partial differential equations, it is necessary to first thoroughly examine the solution to the simplest linear first order partial differential equation. Non-linear PDE –

$$u_t + u u_x = 0 \qquad \ldots\ldots (5)$$

Is known as non-linear transport equation.

Poisson and Riemann were the first to extensively investigate this phenomenon in the early 19th century. Due to its widespread applicability, this equation is known by several different names in the literature. These include the Riemann equation, the inviscid Burgers' equation, and the dispersion-less Korteweg-deVries equation. It and its generalizations to higher dimensions and more components are crucial in the modeling of phenomena as diverse as gas dynamics, traffic flow, river flood waves, chromatography, and chemical reactions *(J. C. Butcher 2008).*

3. RESULT:

To explore the various solutions of nonlinear partial differential equations via some familiar technique we introduce this part.

3.1. BASIC IDEA:

It's a standard approach to solving non-linear PDEs by determining their full integral,

$$f(x, y, z, p, q) = 0 \quad \ldots\ldots (6)$$

The fundamental concept behind this approach is to add another PDE –

$$g(x, y, z, p, q, a) = 0 \ldots\ldots\ldots (7)$$

This has an undefined constant called a and is such that

Equation (6) and (7) is solvable for p and q to obtain, $p = p(x, y, z, a)$, $q = q(x, y, z, a)$

The equation $dz = p(x, y, z, a)dx + q(x, y, z, a)dy$ ......(8), can be integrated.

If such a function g is discovered, the solution, $F(x, y, z, a, b) = 0$

of (8) consisting of two arbitrary constant a , b will be the solution of (6).

3.2. COMPLETE SOLUTIONS OF NON-LINEAR PARTIAL DIFFERENTIAL:

Let us consider a non-linear partial differential equation (PDE)

$$f(x, y, z, p, q) = 0 \ldots\ldots\ldots\ldots (9)$$

The universal solving approach of the equation (9) entails locating an appropriate equation.

$$F(x, y, z, p, q) = 0 \quad \ldots\ldots\ldots (10)$$

Where $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$ such that (9) and (10) may solving for the derivatives p and q .

Since z is a function of the variables x and y , then it follows that there exists a relations $dz = pdx + qdy$; should be integrable if $\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0$

Now taking the partial differentiation of (1) w.r.to x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0$$

$$f_x + pf_z + f_p p_x + q_x f_q = 0 \quad \ldots\ldots\ldots (11)$$

And

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial y} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial y} = 0$$

$$f_y + qf_z + f_p p_y + q_y f_q = 0 \quad \ldots\ldots\ldots (12)$$

Similarly, taking the partial differentiation w.r.to x and y, we get

$$F_X + pF_z + F_P p_x + F_q q_x = 0 \quad \ldots\ldots\ldots\ldots\ldots (13)$$

And

$$F_y + qF_z + F_p p_y + F_q q_y = 0 \quad \ldots\ldots\ldots\ldots\ldots (14)$$

Multiplying (11) byF_p, (12) by F_q , (13) by -f_p , (14) by -f_q and then adding all terms , we get

$$F_p f_x + pF_p f_z + F_p f_p p_x + F_P f_q q_x + F_q f_y + qF_q f_z + F_q f_p p_y + F_q f_q q_y - f_p F_x - pf_p F_z -$$

$$f_p F_p p_x - f_p F_q q_x - f_q F_y - qf_q F_z - f_q F_p p_y - f_q F_q q_y = 0$$

$$\Rightarrow \quad (f_x + pf_z)F_p + (f_y + qf_z)F_q - f_pF_x - f_qF_y - (pf_p + qf_q)F_z = 0$$

This is a linear partial differential equation in F. Then it has the auxiliary system i.e,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

This is the required auxiliary system from which , we can determine the solution for p and q This system is called charpit's method.

3.1 EXAMPLE:

$$p^2 - x = q^2 - y$$

SOLUTION:

Let f(x , y , z , p , q) $= p^2 - q^2 - x + y$

$\therefore f_x = -1, f_y = 1, \quad f_z = 0 \quad , \quad f_p = 2p \quad , \quad f_q = -2q$

We know the Charpit's auxiliary system is

$$\frac{dp}{f_{x+pf_z}} = \frac{dq}{f_{y+qf_z}} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-(pf_p + qf_q)}$$

$$\Rightarrow \frac{dp}{-1+o} = \frac{dq}{1+0} = \frac{dx}{-2p} = \frac{dy}{2q} = \frac{dz}{-(2q^2 - 2q^2)}$$

$$\Rightarrow \frac{dp}{-1} = \frac{dq}{1} = \frac{dx}{-2p} = \frac{dy}{2q} = \frac{dz}{-2q^2 + 2q^2}$$

From first and third ratio of (i) , we get

$$\frac{dp}{-1} = \frac{dq}{1} \Rightarrow \quad 2p\,dp = dx$$

Integrating this, we get , $p^2 = x + a$ or $p = \sqrt{x+a}$

Now from second and fourth ratio of (i), we get

$$\frac{dp}{-1} = \frac{dq}{1} \Rightarrow \quad 2q\,dq = dy$$

Integrating this, we get

$q^2 = y + b \Rightarrow q = \sqrt{(y+b)}$ Finally we use the result of p and q in the relation

dz = p dx + q dy , $dz = \sqrt{x+a}\,dx + \sqrt{y+b}\,dy$

Integrating this, we get

$$z = \tfrac{2}{3}\sqrt[3]{x+a} + \tfrac{2}{3}\sqrt[3]{y+b} + \tfrac{2}{3}c$$

$$z = \tfrac{2}{3}\left[\sqrt[3]{x+a} + \sqrt[3]{y+b} + c\right]$$

This is the required solution.

4. DISCUSSION:

In order to investigate the potential answers to the nonlinear PDE, we make use of the first integral approach. The theory of commutative algebra underpins this approach to problem solving. Using the first integral method,

Think about the following systems of PDE:

$$\Psi_1(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, \ldots\ldots) = 0, \quad \ldots (15)$$

$$\Psi_2(u\,,v\,,u_t\,,\ v_t\,,u_x\,,v_x\,,u_{tt}\,,v_{tt}\,,u_{xx}\,,v_{xx}\,,\dots\dots) = 0\,, \quad \dots.(16)$$

We make use of transformations.

$$u(x\,,t) = f(\xi)\,,v(x\,,t) = g(\xi)\ ,\xi = x - ct. \qquad \dots\dots.(17)$$

Using Equation $\frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.)$, to move the systems of NPDE (15), (16) in terms of systems of ODE

$$\Gamma_1(\,f\,,g\,,f'\,,g'\,,\dots.. = 0\,, \qquad \dots\dots..(18)$$

$$\Gamma_2(\,f\,,g\,,f'\,,g'\,,\dots.. = 0\,, \qquad \dots\dots..(19)$$

Using some mathematical operations, the systems of ODE (31) and (32) is transformed into a second-order ordinary DE as

$$\Omega(f\,,f'\,,f''\,,\dots\ \dots)b = 0\,. \qquad \dots\dots..\ (20)$$

If we consider $X(\xi) = f(\xi)\,,\ Y(\xi) = f'(\xi)$, the Equation (20) is corresponds to the two dimensional self-explanatory system

$$\begin{cases} X' = Y\,, \\ Y' = \Delta(X\,,Y). \end{cases} \qquad \dots\dots..(21)$$

To begin, we will use the Division Theorem to obtain one first integral to Equation (21), reducing Equation (20) to a first order integrable ODE. Solving this equation yields an exact solution to NPDE systems (15) and (16).

$$\begin{cases} u_t - uu_x - v_x + \alpha u_{xx} = 0 \\ v_t - (\,uv\,)_x - \alpha v_{xx} = 0. \end{cases} \qquad \dots\dots\dots..(22)$$

Making the transformation

$$u(x\,,t) = u(\xi)\,,v(x\,,t) = v(\xi)\ ,\xi = kx - lt \quad ,$$

We alter the ALWW system (22) to the pursuing system of ODE

$$\begin{cases} lu' - kuu' + kv' + \alpha k^2 u'' = 0, \\ lv' - k(\,uv\,)' - \alpha k^2 v'' = 0\,. \end{cases} \qquad \dots\dots\dots.(23)$$

By integrating the first equation we have

$$lu - \frac{k}{2}u^2 - kv + \alpha k^2 u' = R_1\,, \qquad \dots\dots\ \dots..(24)$$

Where R_1integration constant, this equation is should be rewritten as follows:

$$v(\xi) = \frac{l}{k}u - \frac{u^2}{2} + \alpha k u' - \frac{R_1}{k}. \qquad \dots\dots\dots..(25)$$

Equation (25) is inserted into the second system (22) and then integrated to produce $\left(\frac{l^2}{k} + R_1\right)u - \frac{3l}{2}u^2 + \frac{k}{2}u^3 - \alpha^2 k^3 u'' = R_2\,, \qquad \dots\dots\dots.(26)$

Where R_2 is integration constant?

If we consider $X = f(\xi), Y = \frac{df(\,\xi\,)}{d\xi}$, the equation (26) is corresponds to the two dimensional self-explanatory system

$$\begin{cases} X' = Y\,, \\ Y' = \left(\frac{1}{2\alpha^2 k^2}\right)X^3(\xi) - \frac{3l}{2\alpha^2 k^3}X^2(\xi) + \left(\frac{l^2}{\alpha^2 k^4} + \frac{R_1}{\alpha^2 k^3}\right)X(\xi) - \frac{R_2}{\alpha^2 k^3}. \end{cases} \qquad \dots(27)$$

$$q\big(X(\xi), Y(\xi)\big) = \sum_{i=0}^{m}(a_i\big(X(\xi)\big)Y^i(\xi) = 0\,, \qquad \ldots\ldots (28)$$

is said to be the first integral to equation (27). According to the Division Theorem, there lie a polynomial

$$g(X) + h(X)Yinc[X\,, Y] \quad , \qquad \ldots\ldots .(29)$$

Since $a_i(X)(i = 0\,,1)$ are polynomials, then we deduce that

$a_1(X) = 1$ Constant and $h(x) = 0$. For simplicity, take $a_1(\,x\,)$ Balancing the degrees of g(x) and $a_0(X)$, we conclude that

deg(g (X)) = 1 only. Suppose that $g(X) = B_0 + A_1X$, then we find

$a_0(X)$.

$$a_0(X) = \frac{1}{2}A_1X^2 + B_0X + A_0\,, \qquad \ldots\ldots\ldots\ldots (30)$$

Where A_0 is an arbitrary constant? Set aside $a_0(X), a_1(X)$ and g(X) in the last equation and when we set all the coefficients of powers X to zero, we get a system of nonlinear algebraic equations, which we can solve.

$$B_0 = -\frac{l}{k^2\alpha}\ , A_1 = \frac{1}{k a}\ , R_1 = k^2\alpha A_0\ , R_2 = l\alpha k A_0\,, \qquad \ldots\ldots (31)$$

Where k, l and A_0 are integrating constant.

Apply the postulate (31), we get

$$Y(\xi) = -A_0 + \frac{l}{\alpha k^2}X(\xi) - \frac{1}{2\alpha k}X^2(\xi) \qquad \ldots\ldots\ldots\ldots (32)$$

Combining (32) with (27), We discover the exact answer to equation (26), and the ALWW system's exact solution (25), which can be expressed as

$$\begin{cases} u(x,t) = \dfrac{l}{k} - \dfrac{\sqrt{2\alpha k^3 A_0 - l^2}}{k} tan\left[\dfrac{\sqrt{2\alpha k^3 A_0 - l^2}}{2k^2\alpha}(kx + lt + \xi_0)\right], \\ v(x,t) = \left(\dfrac{l^2}{2k^2} - \dfrac{2\alpha k^3 A_0 - l^2}{2k^2} - \alpha k A_0\right) - \dfrac{2\alpha k^3 A_0 - l^2}{k} tan^2\left[\dfrac{\sqrt{2\alpha k^3 A_0 - l^2}}{2k^2\alpha}(kx + lt + \xi_0\right], \end{cases}$$

For $2\alpha k^3 A_0 > l^2$.

This technique performs well, is dependable, and offers a variety of solutions. The method is advantageous because it is straightforward and concise.

4. CONCLUSION:

Nonlinear partial differential equations, a broad and lively area, will be the work final point of focus. All efforts to summarize the full breadth of mathematical phenomena, methods, conclusions, and developments in a single volume (or even a large one) are doomed to failure. Since there are so many important physical and mathematical phenomena associated with nonlinear partial differential equations, we have been content to introduce only a small number of topical, seminal instances that occur in the study of nonlinear partial differential equations.

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