INTERNATIONAL JOURNAL FOR INNOVATIVE RESEARCH IN MULTIDISCIPLINARY FIELD
ISSN(O): 2455-0620 [Impact Factor: 9.47]
Monthly, Peer-Reviewed, Refereed, Indexed Journal with IC Value : 86.87
Volume - 12, Issue - 1, January - 2026

DOIs:10.2015/IJIRMF/202601015 --:-- Research Paper / Article / Review

Local stability of a fractional order Leslie-gower type model with Holling type IV functional response on intermediate predator

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Abstract: *The predator-prey interaction is one of the most powerful forces shaping an ecosystem. The dynamic relationship between a predator and its prey, particularly when a top predator exists, has profound and cascading impacts on the entire ecosystem. In summary, the health of an ecosystem is often dependent on the presence and stability of its top predator population, as they provide the top-down structure necessary to prevent one species from dominating and to maintain ecological complexity. Recently, the dynamical behaviors of a fractional order three species food chain model with a simplified Holling type IV functional response was studied by S.Mondal by incorporating memory effect (Applied Mathematical Biosystems, 1(1),10-21, 2025). He shown that the fractional order system shows more complex dynamics, like chaos, bifurcation for lower memory as the fractional order becomes larger and shows more simpler dynamics for higher memory as the fractional order decreases. Some qualitative behaviors like existence, uniqueness, non-negativity and boundedness are discussed in a feasible domain except the dissipativeness of the solutions . He also stated the local and global stability criteria of the interior equilibrium point but however, the proof of local stability criterion was left over. This work extends his work and gives the proof of the local stability criterion of the interior equilibrium point. Dissipativeness of the solutions of fractional order system is also proved in a feasible domain. Numerical examples are also provided to substantiate the analytical findings.*

Key Words: *Caputo fractional order derivative, Food Chain model, Holling type IV functional response, Local stability , Periodic solution, Limit cycle, Bifurcation.*

1. INTRODUCTION:

Fractional calculus deals with the study of fractional-order integral and derivative operators over real or complex domains and their applications.[1] In ecological systems, fractional order models have also been used to understand the dynamics of interacting populations.[2-10] Lots of researchers are also showing interest to study the dynamics of discretized fractional-order systems and able to find more complex behaviors depending on both the step-size and fractional-order.[11-12] Due to the presence of memory, fractional derivative find wider area of applications and the models involving fractional derivatives provide a better agreement with real data than integer order derivative models. The value of the index of the fractional derivative, m (say), characterizes the way in which the memory along different parts of the interval of integration affects the solution at a given time and it can be varied to best fit the real data.[13-14] Therefore, the influence of memory concepts on several dynamical systems gained more popularity to the young researchers.[15-17]

After the seminal work of May[18], exploring the chaotic behaviors in population models became a fascination. Lot of mathematical models have been proposed on food-chain and analyzed to show complicated dynamics like chaos.[19-25] Aziz-Alaoui discussed the complex dynamics in a modified Leslie-Gower three species food chain model with Holling type II response function.[26] In, Naji et al. also discussed about the chaotic dynamics in a modified Leslie-Gower food chain model with Bendigton-DeAnglis functional response.[22] Gakkhar with Priyadarshi proposed a Leslie-Gower food cahin model.[23] The functional response of Holling type IV describes a situation in which the predator's per capita rate of predation decreases at sufficiently high prey densities. Upadhyay used Holling type IV functional response to investigate the existence of complex dynamics in a three species food chain model.[27] For details about this functional response, readers are referred to see Holling work in 1965.[28] Sokol and Howell in suggested a simplified Holling type

IV function and found that it is simpler and better than the original function of Holling type IV.[29] The more interesting fact is that they have all studied the integer order three species food chain models with this type of functional response.

In recent past, Alidousti and Ghahfarokhi [30] extended the work of Aziz-Alaoui [26] and analyzed the following fractional order tri-trophic food chain model with Holling type II functional response:

$$\begin{aligned} {}_{0}^{c}D_{T}^{\alpha}X &= X - \frac{\epsilon}{\delta}X^{2} - \frac{v_{0}XY}{\delta(d_{0} + X)}, X(0) \geq 0, \\ {}_{0}^{c}D_{T}^{\alpha}Y &= -\frac{g}{\delta}Y + \frac{v_{1}XY}{\delta(d_{1} + X)} - \frac{v_{2}YZ}{\delta(d_{2} + Y)}, Y(0) \geq 0, \qquad (1) \\ {}_{0}^{c}D_{T}^{\alpha}Z &= \frac{f}{\delta}Z^{2} - \frac{v_{3}Z^{2}}{\delta(d_{3} + Y)}, Z(0) \geq 0, \end{aligned}$$

where X, Y, Z are, respectively, the densities of prey, intermediate predator and top predator at any instant of at time T. Here, ${}_{0}^{c}D_{T}^{\alpha}X = \frac{d^{\alpha}}{dT^{\alpha}}$ is the operator of the Caputo type fractional order derivative at T with fractional order α ($0 < \alpha < 1$).[31] Sambath in also studied the asymptotic behavior of a fractional order three species predator-prey model with the same functional response.[32] Investigations in fractional order Leslie-Gower type model with Holling type IV functional response is relatively less studied in population ecology. Therefore, in this paper, I consider a three species food chain model with simplified Holling type IV functional response to understand underlying dynamics of the model with respect to fractional order. Recently, Ali et. al.[33] studied the following three-dimension coupled nonlinear autonomous system of integer order differential equations with non-monotone functional response (also called simplified Holling type IV functional response) to understand the underlying dynamics of food chain model:

$$\begin{aligned} \frac{dX}{dT} &= a_{0}X - b_{0}X^{2} - \frac{v_{0}XY}{(d_{1} + X^{2})}, X(0) \geq 0, \\ \frac{dY}{dT} &= -a_{1}Y + \frac{v_{1}XY}{(d_{1} + X^{2})} - \frac{v_{2}YZ}{(d_{2} + Y)}, Y(0) \geq 0, \qquad (2) \\ \frac{dZ}{dT} &= c_{3}Z^{2} - \frac{v_{3}Z^{2}}{(d_{3} + Y)}, Z(0) \geq 0, \end{aligned}$$

where X, Y, Z are, respectively, the densities of prey, intermediate predator and top predator at any instant of at time T. This model considers interactions between a generalist top predator, specialist middle predator, and prey. Here, the specialist middle predator is consumed by the top predator, at a Holling type II rate. The interactions between the specialist middle predator and prey are modeled via a modified Holling type IV functional response. The interaction between the generalist top predator and specialist middle predator follow a modified Leslie-Gower scheme. That is the generalist top predator grows quadratically, because of sexual reproduction as $c_{3}Z^{2}$, and loses because of intra-species competition as$-\frac{v_{3}Z^{2}}{(d_{3}+Y)}$. The d_{3} signifies that Z is a generalist. The biological interpretation of all the parameters are described in the following:

Table 1 Parameter interpretation

Symbol	Meaning
a_0	Growth rate of prey
b_0	Intra specific competition coefficient
$v_{i's}$	Maximum values that per-capita rate can attain
d_1	Measure of protection level provided by the environment to the prey
a_1	Death rate of intermediate predator
d_2	Half-Saturation constant
c_3	Growth rate of top predator via sexual reproduction
d_3	Residual loss of top predator due to severe scarcity of it's favorite prey, Y

All parameters are non-zero positive.

With the transformations

$$X = \frac{a_0}{b_0}x, Y = \frac{a_0^2}{b_0 v_0}y, Z = \frac{a_0^3}{b_0 v_0 v_2}z, \qquad T = \frac{t}{a_0},$$

the system (2) takes the simplified form

$$\begin{aligned}
\frac{dx}{dt} &= x - x^2 - \frac{sxy}{(x^2 + a)}, x(0) = x_0 \geq 0,\\
\frac{dy}{dt} &= -by + \frac{cxy}{(x^2 + a)} - \frac{yz}{(y + d)}, y(0) = y_0 \geq 0, \qquad (3)\\
\frac{dz}{dt} &= pz^2 - \frac{qz^2}{y + r}, z(0) = z_0 \geq 0.
\end{aligned}$$

where

Starting from the integer-order three species Leslie~Gower type food chain model presented by (2), I introduce the Caputo-type fractional order derivatives by replacing the usual integer-order derivatives to obtain the following fractional order system:

$$\begin{aligned}
{}_0^cD_T^m X &= a_0 X - b_0 X^2 - \frac{v_0 XY}{(d_1 + X^2)}, X(0) \geq 0,\\
{}_0^cD_T^m Y &= -a_1 Y + \frac{v_1 XY}{(d_1 + X^2)} - \frac{v_2 YZ}{(d_2 + Y)}, Y(0) \geq 0, \qquad (4)\\
{}_0^cD_T^m Z &= c_3 Z^2 - \frac{v_3 Z^2}{(d_3 + Y)}, Z(0) \geq 0,
\end{aligned}$$

with the initial conditions $X(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0$,, where ${}_0^cD_T^m X = \frac{d^m}{dT^m}$ is the Caputo fractional derivative at T with fractional order $m(0 < m < 1)$.[31] With the same transformations as before, the system (4) takes the following simplified form :

$$\begin{aligned}
{}_0^cD_t^m x &= x - x^2 - \frac{sxy}{(x^2 + a)}, x(0) = x_0 \geq 0,\\
{}_0^cD_t^m y &= -by + \frac{cxy}{(x^2 + a)} - \frac{yz}{(y + d)}, y(0) = y_0 \geq 0, \qquad (5)\\
{}_0^cD_t^m z &= pz^2 - \frac{qz^2}{y + r}, z(0) = z_0 \geq 0.
\end{aligned}$$

The state space of the system (5) is the positive cone $\mathbb{R}_+^3 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$. Mondal has shown that the solutions of system (5) are exists uniquely and positively invariant in $\mathbb{R}_+^3$ under some restrictions.[34] He also proved that the solutions of system (5) are uniformly bounded in $\mathbb{R}_+^3$ under some conditions. Stability conditions for several equilibrium points are also discussed. But, the dissipativeness of the solutions of the fractional order systems (5) was not discussed before. The proof of local stability analysis of the coexistence (or interior) equilibrium point, however, was also omitted as in the case of fractional order system (5). I here extend the work of Mondal[34] by proving the local stability criteria of the interior equilibrium point for both the integer and fractional order systems with the help of Routh-Hurwitz criterion for fractional order differential equations. Dissipativeness of the solutions of the fractional order system (5) is also proved in $\mathbb{R}_+^3$. Simulation results are also given to validate the analytical results.

2. MATHEMATICAL RESULTS:

Mondal[34] proved the following results regarding existence, uniqueness, positivity and boundedness of the solutions of system (5). Here I prove the dissipativeness of the system (5) in following theorem.

Theorem 1: *All the non negative solutions of system (5) which are initiating in* $\mathbb{R}_+^3$ *are uniformly bounded for* $0 < m \leq 1$, *provided*

$$\beta + \frac{\beta}{4b} + r < \frac{q}{p} \qquad (6)$$

and ultimately entering the region

$$\Omega = \{(x,,y,z) \in \mathbb{R}^3_+ : 0 \le x \le 1, 0 \le x + \frac{y}{\beta} \le 1 + \frac{1}{4b}, 0 \le x + \frac{y}{\beta} + \alpha z \le 1 + \frac{1}{4b} + \frac{M}{b}\},$$

where

$$\beta = \frac{v_1}{a_0}, \alpha = \frac{1}{b^2(\beta + \frac{\beta}{4b} + r)}, M = \frac{1}{4(q - (\beta + \frac{\beta}{4b} + r)p)}.$$

Moreover, the system (5) is dissipative in $\mathbb{R}^3_+$.

Proof: Mondal has proved the first part of the above theorem for the fractional order system (5).[34] Here, I will show the dissipativeness of the system (5) in next. From biological point of view, dissipativeness means all populations are bounded above. To prove this, I have to calculate the supremum of $x(t), V_1(t) = x(t) + \frac{y(t)}{\beta}$ and $V_2(t) = x(t) + \frac{y(t)}{\beta} + \alpha z(t)$ as $t \to +\infty$ for $0 < m \le 1$. The steps are following:

$\boldsymbol{Step(i-a)}$: $\lim_{t\to+\infty} \sup x(t) \le 1;$

$\boldsymbol{Step(i-b)}$: $\lim_{t\to+\infty} \sup V_1(t) \le 1 + \frac{1}{4b};$

$\boldsymbol{Step(i-c)}$: $\lim_{t\to+\infty} \sup V_2(t) \le 1 + \frac{1}{4b} + \frac{M}{b}.$

Proof of Step(i-a): Following the first part of this theorem, since any non-negative solution $(x(t), y(t), z(t))$of (5) satisfies $x(t) \le 1, \forall t \ge 0$, so clearly for any $m \in (0,1]$, $\lim_{t\to+\infty} \sup x(t) \le 1$.

Proof of Step(i-b): For the calculation of $\lim_{t\to+\infty} \sup V_1(t); V_1(t) = x(t) + \frac{y(t)}{\beta}$, let $\epsilon > 0$ be given. Then there exists a t_1>0 such that $x(t) \le 1 + \frac{\epsilon}{2}, \forall t \ge t_1$. Applying $0 \le x \le 1$ and $max_{[0,1]} x(1-x) = \frac{1}{4}$ for all $t \ge t_1 \ge 0$, equation (18) in Mondal[34] gives,

$$\begin{aligned} V_1(t) &\le 1 + \frac{1}{4b} - \left[1 + \frac{1}{4b} - (x(t_1) + \frac{y(t_1)}{\beta})\right] E_m[-b(t-t_1)^m)], \\ &= 1 + \frac{1}{4b} - \left[1 + \frac{1}{4b} - (x(t_1) + \frac{y(t_1)}{\beta})\right] e^{-b(t-t_1)^m}, \\ &\le 1 + \frac{1}{4b} - \left[1 + \frac{1}{4b} - (x(t_1) + \frac{y(t_1)}{\beta})\right] e^{-b(t-t_1)}, [\text{as } 0 < m \le 1] \\ &= 1 + \frac{1}{4b} - \left[(1 + \frac{1}{4b}) e^{bt_1} - (x(t_1) + \frac{y(t_1)}{\beta})e^{bt_1}\right] e^{-bt}, \\ &\le 1 + \frac{1}{4b} - \left[(1 + \frac{1}{4b}) - (x(t_1) + \frac{y(t_1)}{\beta})e^{bt_1}\right] e^{-bt}, \\ &\le (1 + \frac{1}{4b} + \frac{\epsilon}{2}) - \left[(1 + \frac{1}{4b} + \frac{\epsilon}{2}) - (x(t_1) + \frac{y(t_1)}{\beta})e^{bt_1}\right] e^{-bt}, \end{aligned}$$

For all $t \ge t_1$. Suppose $t_2 \ge t_1$ be such that

$$\left| (1 + \frac{1}{4b} + \frac{\epsilon}{2}) - \left[(1 + \frac{1}{4b} + \frac{\epsilon}{2}) - (x(t_1) + \frac{y(t_1)}{\beta})e^{bt_1}\right] e^{-bt} \right| \le \frac{\epsilon}{2}, \text{for all } t \ge t_2$$

Then I get

$$V_1(t) \le 1 + \frac{1}{4b} + \epsilon \text{ for all } t \ge t_2 .$$

Hence, $\lim_{t\to+\infty} \sup V_1(t) \le 1 + \frac{1}{4b}$; for any $m \in (0,1]$.

Proof of Step(i-c): Similarly I can consider $\epsilon > 0$. Then thre exists a t_3>0 such that $V_2(t) \le 1 + \frac{1}{4b} + \frac{\epsilon}{2}, \forall t \ge t_3$. . Next considering the equation (22) in Mondal[34], I get for all $t \ge t_3 \ge 0$,

$$\begin{aligned}
V_2(t) &\le 1+\frac{1}{4b}+\frac{M}{b}-\left[1+\frac{1}{4b}+\frac{M}{b}-(x(t_3)+\frac{y(t_3)}{\beta}+\alpha z(t_3))\right]E_m[-b(t-t_3)^m)],\\
&= 1+\frac{1}{4b}+\frac{M}{b}-\left[1+\frac{1}{4b}+\frac{M}{b}-(x(t_3)+\frac{y(t_3)}{\beta}+\alpha z(t_3))\right]e^{-b(t-t_3)^m},\\
&\le 1+\frac{1}{4b}+\frac{M}{b}-\left[1+\frac{1}{4b}+\frac{M}{b}-(x(t_3)+\frac{y(t_3)}{\beta}+\alpha z(t_3))\right]e^{-b(t-t_3)}, [\text{as } 0< m \le 1]\\
&= 1+\frac{1}{4b}+\frac{M}{b}-\left[(1+\frac{1}{4b}+\frac{M}{b})\, e^{bt_3}-(x(t_3)+\frac{y(t_3)}{\beta}+\alpha z(t_3))e^{bt_3}\right]e^{-bt},\\
&\le 1+\frac{1}{4b}+\frac{M}{b}-\left[(1+\frac{1}{4b}+\frac{M}{b})-(x(t_3)+\frac{y(t_3)}{\beta}+\alpha z(t_3))e^{bt_3}\right]e^{-bt},\\
&\le (1+\frac{1}{4b}+\frac{M}{b}+\frac{\epsilon}{2})-\left[(1+\frac{1}{4b}+\frac{M}{b}+\frac{\epsilon}{2})-(x(t_3)+\frac{y(t_3)}{\beta}+\alpha z(t_3))e^{bt_3}\right]e^{-bt},
\end{aligned}$$

For all $t \ge t_3$. Suppose $t_4 \ge t_3$ be such that

$$\left|\,(1+\frac{1}{4b}+\frac{M}{b}+\frac{\epsilon}{2})-\left[(1+\frac{1}{4b}+\frac{M}{b}+\frac{\epsilon}{2})-(x(t_3)+\frac{y(t_3)}{\beta}+\alpha z(t_3))e^{bt_3}\right]e^{-bt}\right|\le\frac{\epsilon}{2}, \text{for all } t \ge t_4$$

.

Then I get

$$V_2(t) \le 1+\frac{1}{4b}+\frac{M}{b}+\epsilon \text{ for all } t \ge t_4 .$$

Hence, $\lim\limits_{t\to+\infty}\sup V_2(t) \le 1+\frac{1}{4b}+\frac{M}{b}$; for any $m \in (0,1]$. Therefore I can say that our system (5) is dissipative in $\mathbb{R}_+^3$ for all $m \in (0,1]$. This completes the proof of the theorem.

3. EXISTENCE AND STABILITY OF EQUILIBRIA:

I have the following stability result on fractional order dynamical systems.[35-56]

Theorem 2: Consider the following fractional order system

$${}_0^cD_t^m x(t) = f(x), x(0) = x_0$$

with $0 < m < 1, x \in \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$. The equilibrium points of the above system are calculated by solving the equation: $f(x) = 0$. These equilibrium points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix $J = \frac{\delta f}{\delta x}$ evaluated at the equilibrium points satisfy $|Arg(\lambda_i)| > \frac{m\pi}{2}, i = 1,2,3,\dots\dots.n.$

An equilibrium point of system (5) is found by solving the three equations ${}_0^cD_t^m x(t) = {}_0^cD_t^m y(t) = {}_0^cD_t^m z(t) = 0$ in (5). There are four biologically feasible non-negative equilibrium points of system (5). The trivial equilibrium $E_0 = (0,0,0)$and the axial equilibrium $E_1 = (1,0,0)$ are always exist. The planner equilibrium point $E_2 = (\bar{x},\bar{y},0)$ exists uniquely in the positive quadrant of xy plane, where $\bar{x} = \frac{c}{2b}, \bar{y} = \frac{1}{s}(1-\bar{x})(a+\bar{x}^2)$, provided that the following conditions are hold

$$\frac{c}{2b} < 1, c^2 - 4ab^2 = 0. \qquad (8)$$

I observe that in the absence of prey x, both predators y and z can not survive. So there is no equilibrium point in the yz −plane. Similarly I can also conclude that there is no equilibrium point in xz- plane. Now there exists a unique interior equilibrium point $E^* = (x^*, y^*, z^*)$ of the system (5), where the equilibrium population densities are given by

$$y^* = \frac{q}{p} - r, \qquad (9)$$

while $x^*$is the positive root of the cubic equation

$$x^3 - x^2 + ax + (sy^* - a) = 0, \qquad (10)$$

this equation can be written as

$$f(x) = Ax^3 + Bx^2 + Cx + D = 0, \qquad (11)$$

where $A = 1, B = -1, C = a$ and $D = (sy^* - a)$. Now since $0 \le x^* \le 1$, then $f(0) = D > 0$, if $y^* = \frac{a^*}{s}$

and $\mathrm{f}(1) = \mathrm{sy}^* > 0$. Thus $\mathrm{f}(0)\mathrm{f}(1) = \mathrm{sy}^*(\mathrm{sy}^* - \mathrm{a}) < 0$, then there is positive root of equation (11) lies in (0,1) when $y^* < \frac{a}{s}$ is satisfied. Now from the second equation of system (5), I obtain

$$z^* = (-b + \frac{cx^*}{a + x^{*2}})(y^* + d), \qquad (12)$$

and it exists if $b < \frac{cx^*}{a + x^{*2}}$. Therefore the positivity condition of E^* in $\mathbb{R}^3_+$ are

$$y^* < \frac{a}{s}, b < \frac{cx^*}{a + x^{*2}}, where\ v_3 > c_3 d_3.$$

Different stability results for the equilibrium points E_0, E_1, E_2 and E^* are given in the following. Now to investigate the dynamical behavior of the equilibrium points E_0, E_1, E_2 and E^*, I first construct the Jacobian matrix J evaluated at an equilibrium point (x, y, z) of the system (5) is

$$\mathrm{J}(\mathrm{x},\mathrm{y},\mathrm{z}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad (13)$$

Where

$$a_{11} = 1 - 2x - \frac{sy(a - x^2)}{(a + x^2)^2},$$

$$a_{12} = -\frac{sx}{(a + x^2)}, a_{13} = 0,\ a_{21} = \frac{cy(a - x^2)}{(a + x^2)^2}, a_{22} = -b + \frac{cx}{(a + x^2)} - \frac{dz}{(d + y)^2}, a_{23} = -\frac{y}{(d + y)},\quad a_{31} = 0, a_{32} = \frac{qz^2}{(r + y)^2}, a_{33} = 2z(p - \frac{q}{(r + y)}).$$

Then the Jacobian matrices evaluated at E_0, E_1, E_2 are given by

$$\mathrm{J}(\mathrm{E}_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{J}(\mathrm{E}_1) = \begin{pmatrix} -1 & -\frac{s}{a + 1} & 0 \\ 0 & -b + \frac{c}{a + 1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{J}(\mathrm{E}_2) = \begin{pmatrix} 1 - 2\bar{x} - \frac{(1 - \bar{x})(a - \bar{x}^2)}{a + \bar{x}^2} & -\frac{s\bar{x}}{a + \bar{x}^2} & 0 \\ \frac{c(1 - \bar{x})(a - \bar{x}^2)}{s(a + \bar{x}^2)} & 0 & -\frac{\bar{y}}{d + \bar{y}} \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly, the eigenvalues of $J(E_0)$ are $\xi_1 = 1, \xi_2 = -b\ and\ \xi_3 = 0$. Note that $\mathrm{Arg}(\xi_3)$ is undefined. Since one of them is a positive real and another one is a negative real, then E_0 is always unstable. Therefore E_0 is non-hyperbolic.

Next, the eigenvalues of $J(E_1)$ are $\xi_1 = -1 < 0, \xi_2 = \frac{c-b-ab}{1+a}\ and\ \xi_3 = 0$. Hence E_1 is also non-hyperbolic. Note that if $\mathrm{c} - \mathrm{b} > \mathrm{ab}$ then $\xi_2 > 0$. In this case, E_1 is always unstable saddle along x- direction. If $\mathrm{c} - \mathrm{b} < \mathrm{ab}$, then $\xi_2 < 0$. Consequently, two of the eigenvalues are negative real, so in this case E_1 is stable manifold along x and y direction. Again from the Jacobian matrix E_2 , the eigenvalues $J(E_1)$ are $\xi_{1,2} = \frac{1}{2}[P \pm \sqrt{(P^2 - 4Q)}]$, where $\mathrm{P} = 1 - 2\bar{x} - \frac{(1 - \bar{x})(a - \bar{x}^2)}{a + \bar{x}^2}$, $\mathrm{Q} = \frac{c\bar{x}(1 - \bar{x})(a - \bar{x}^2)}{(a + \bar{x}^2)^2}$ and $\xi_3 = 0$. Since one of the eigenvalue ξ_3 becomes zero, so E_2 is non-hyperbolic equilibrium point.

For local stability of the interior equilibrium E^*, I compute the Jacobian matrix of system (5) at $E^* = (x^*, y^{*,} z^*)$ as

$$J(E^*) = \begin{pmatrix} 1 - 2x^* - \dfrac{(1 - x^*)(a - x^{*2})}{a + x^{*2}} & -\dfrac{sx^*}{a + x^{*2}} & 0 \\ \dfrac{c(1 - x^*)(a - x^{*2})}{s(a + x^{*2})} & \dfrac{y^*z^*}{(y^* + d)^2} & -\dfrac{y^*}{(y^* + d)} \\ 0 & \dfrac{pz^{*2}}{(y^* + r)} & 0 \end{pmatrix} \qquad (14)$$

The eigenvalues are the roots of the cubic equation $F(\xi) = \xi^3 + A_1\xi^2 + A_2\,\xi + A_3 = 0,$ (15)
Where

$$A_1 = -1 + 2x^* + \frac{(1 - x^*)(a - x^{*2})}{a + x^{*2}} - \frac{y^*z^*}{(y^* + d)^2},$$

$$A_2 = \frac{y^*z^*}{(y^* + d)^2}\left(1 - 2x^* - \frac{(1 - x^*)(a - x^{*2})}{a + x^{*2}}\right) + \frac{cx^*(1 - x^*)(a - x^{*2})}{(a + x^{*2})^2} + \frac{py^*z^{*2}}{(y^* + d)(y^* + r)},$$

$$A_3 = -\frac{py^*z^{*2}}{(y^* + d)(y^* + r)}\left(1 - 2x^* - \frac{(1 - x^*)(a - x^{*2})}{a + x^{*2}}\right).$$

The equilibrium E^* is said to be locally asymptotically stable if all eigenvalues of (15) satisfy $|Arg(\xi_i)| > \frac{m\pi}{2}, \forall m \in (0,1], i = 1,2,3$. One can then determine the stability of E^* by noting the signs of the coefficients A_i and discriminant $D(F)$ of the cubic polynomial $F(\xi)$, by following Routh-Hurwitz criterion for fractional order differential equations.[37] The discriminant $D(F)$of the cubic polynomial $F(\xi)$ is

$$D(F) = -\begin{vmatrix} 1 & A_1 & A_2 & A_3 & 0 \\ 0 & 1 & A_1 & A_2 & A_3 \\ 3 & 2A_1 & A_2 & 0 & 0 \\ 0 & 3 & 2A_1 & A_2 & 0 \\ 0 & 0 & 3 & 2A_1 & A_2 \end{vmatrix}$$
$$= 18A_1 A_2A_3 + (A_1 A_2)^2 - 4A_3A_1^3 - 4A_2^3 - 27A_3^2$$

.

Then the following theorem regarding local asymptotic stability of E^* of the system (5) is true.[37]

Theorem 3: *(i) If* $D(F) > 0, A_1 > 0, A_3 > 0$ *and* $A_1 A_2 - A_3 > 0$ *then the interior equilibrium* E^* *is locally asymptotically stable for all* $m \in (0,1]$.
(ii) If $D(F) < 0, A_1 \geq 0, A_2 \geq 0, A_3 > 0$ *and* $0 < m < \frac{2}{3}$ *then the interior equilibrium* E^* *is locally asymptotically stable.*
(iii)If $D(F) < 0, A_1 < 0, A_2 < 0$ *and* $m > \frac{2}{3}$ *then the interior equilibrium* E^* *is unstable.*
(iv) If $D(F) < 0, A_1 > 0, A_2 > 0, A_1 A_2 = A_3$ *and* $0 < m < 1$ *then the interior equilibrium* E^**is locally asymptotically stable.*

Proof: If $D(F)$ is positive then all the roots of (15) are real and distinct. If not, let us assume that $F(\xi) = 0$ has one real root $\xi_1$and another two complex conjugate roots ξ_2 and ξ_3. In terms of the roots, the discriminant of $F(\xi)$ can be written as[38]

$$D(F) = [(\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_2 - \xi_3)]^2 \qquad (16)$$

Note that

$$(\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_2 - \xi_3) = (\xi_1 - \xi_2)(\xi_1 - \bar{\xi_2})(\xi_2 - \bar{\xi_2})$$
$$= (\xi_1 - \xi_2)(\xi_1 - \bar{\xi_2})\, 2\mathrm{Im}(\xi_2)i$$
$$= (\xi_1 - \xi_2)\overline{(\xi_1 - \xi_2)}2\mathrm{Im}(\xi_2)i$$
$$= 2|\xi_1 - \xi_2|^2\mathrm{Im}(\xi_2)i$$

Thus

$$\mathrm{D(F)} = [2|\xi_1 - \xi_2|^2 \mathrm{Im}(\xi_2)\mathrm{i}]^2 < 0, \qquad (17)$$

which contradicts the fact that $\mathrm{D(F)} > 0$. Therefore, whenever $\mathrm{D(F)} > 0$ then $\mathrm{F(\xi)} = 0$ has three real distinct roots. Since $A_1 > 0, A_3 > 0$ *and* $A_1 A_2 - A_3 > 0$, all roots of $\mathrm{F(\xi)} = 0$ has negative real roots or complex conjugate roots with negative real parts. As $\mathrm{D(F)} > 0$, so all roots of $\mathrm{F(\xi)} = 0$ are real negative. Consequently, $|Arg(\xi_i)| = \pi > \frac{m\pi}{2}, \forall m \in (0,1], i = 1,2,3$ and the equilibrium E^* is locally asymptotically stable. This completes the proof of (i).

(ii) I have seen in (i) that $\mathrm{F(\xi)} = 0$ has one real and two complex conjugate roots if $\mathrm{D(F)} < 0$. Since $A_3 > 0$, following (15), the real root is negative. I thus consider the roots as $\xi_1 = -b. b \in \mathbb{R}_+$ and $\xi_{2,3} = \beta \pm i\gamma. \beta, \gamma \in \mathbb{R}_+$ and also I can write

$$F(\xi) = (\xi + b)(\xi - \beta - i\gamma)(\xi - \beta + i\gamma).$$

Comparing this with (15), I have $A_1 = b - 2\beta, A_2 = \beta^2 + \gamma^2 - 2b\beta, A_3 = b(\beta^2 + \gamma^2)$. Now $A_1 \geq 0 \Rightarrow b \geq 2\beta$. Noting $\beta^2(\sec\theta)^2 = \beta^2 + \gamma^2$ and $A_2 \geq 0$, I have $(\sec\theta)^2 \geq 4$. Therefore, $\theta = |Arg(\xi)| \geq \frac{\pi}{3}$. Since $0 < m < \frac{2}{3}$, *then* $\theta = |Arg(\xi)| \geq \frac{\pi}{3} > \frac{m\pi}{2}$ holds. Thus, all roots of (15) satisfy the local stability criteria for all values of fractional order $m \in (0, 1]$ and the equilibrium $\mathrm{E}^*$is locally asymptotically stable. This completes the proof of (ii). Proof of (iii) is similar to the proof of (ii) and hence omitted.

Since $D(F) < 0, A_1 > 0, A_2 > 0$, from the previous case, I have the

$$A_1 = b - 2\beta, A_2 = \beta^2 + \gamma^2 - 2b\beta, A_3 = b(\beta^2 + \gamma^2)$$

Note that $A_1 > 0 \Rightarrow b > 2\beta, A_2 > 0 \Rightarrow \beta^2 + \gamma^2 - 2b\beta > 0$ *and* $A_1 A_2 = A_3 \Rightarrow \beta(\beta^2 + \gamma^2 - 2b\beta) = 0$. Then two cases arise:

Case I: If $\beta = 0$ then three roots ξ_1, ξ_2 and $\xi_3$of (15) are $-\mathrm{b}, \pm\mathrm{i}\gamma$. One can see that $|Arg(\xi_1)| = \pi > \frac{m\pi}{2}, \forall m \in (0,1]$ and $|Arg(\xi_{2,3})| = \frac{\pi}{2} > \frac{m\pi}{2}, \forall m \in (0,1]$and therefore the equilibrium E^* is locally asymptotically stable.

Case II: If $\beta^2 + \gamma^2 - 2b\beta = 0$ then I have $\mathrm{b} = \beta, \gamma = 0$.Using it in $b \geq 2\beta$ and $\beta^2 + \gamma^2 > 2b\beta$, I obtain $\mathrm{b} < 0$, which contradicts the assumption $b \in \mathbb{R}_+$.

Thus, if $A_1 > 0, A_2 > 0, A_1 A_2 = A_3$ then one root is real negative and the other two are purely imaginary and therefore $|Arg(\xi_1)| = \pi > \frac{m\pi}{2}, \forall m \in (0,1)$ and $|Arg(\xi_{2,3})| = \frac{\pi}{2} > \frac{m\pi}{2}, \forall m \in (0,1)$, implying local asymptotic stability of E^*. This completes the proof of the theorem.

4. NUMERICAL SIMULATIONS:

In this section, I perform numerical simulations of the fractional order system (4) for different fractional values of $\mathrm{m}, 0 < \mathrm{m} < 1$ also for $\mathrm{m} = 1$. I use Adams-type predictor corrector method (PECE) for the numerical solution of system (4). Specially, It is very useful method to give numerical solutions of both linear and nonlinear FODE.[39-40] I first replace our system (4) by the following equivalent fractional integral equations:

$$\begin{aligned} X(T) &= X(0) + {}_0^cD_T^{-m}[a_0X - b_0X^2 - \frac{v_0XY}{(d_1 + X^2)}], \\ Y(T) &= Y(0) + {}_0^cD_T^{-m}[-a_1Y + \frac{v_1XY}{(d_1 + X^2)} - \frac{v_2YZ}{(d_2 + Y)}], \qquad (18) \\ Z(T) &= Z(0) + {}_0^cD_T^{-m}[c_3Z^2 - \frac{v_3Z^2}{(d_3 + Y)}]. \end{aligned}$$

and then apply the PECE (Predict, Evaluate, Correct, Evaluate) method.

Various examples are presented to illustrate the analytical results obtained in the previous section. Specially, our main objective is to explore the possibility of dynamical behavior of the fractional order system (4) by depending on the sensitive parameter and as well as the fractional order by keeping others parameters unchanged. To understand the effect of fractional order on the system dynamics, I varied m in its range $0 < m < 1$. I also plotted the solutions for $m = 1$, whenever necessary, to compare the solution of fractional order system with that of integer order. In numerical simulations, Initial values are indicated with stars and equilibrium points are denoted by red circles.

Example 1: Here the parameter values are chosen as $b_0 = 0.075, a_1 = 0.105, d_{1,2} = 10, d_3 = 20, v_0 = 1, v_1 = 2, v_2 = 0.405, v_3 = 1$ and the initial condition (1.2, 1.2, 1.2). All the parameters are taken from Mondal.[34] The bifurcation diagram with respect to sensitive parameters a_0 and c_3 is shown in Fig. 1 for different fractional order $m = 0.95, 0.75$ and the standard order $m = 1$. For the standard orderm $= 1$, it is observed that the system (4) approaches to chaos via period doubling bifurcation for $a_0 \in (0.25, 0.5)$and $c_3 = 0.047$(see Fig. 1(a)). It is interesting to note that the bifurcation disappears slowly with the decreasing of fractional order m (see Figs. 1(b) and 1(c)).

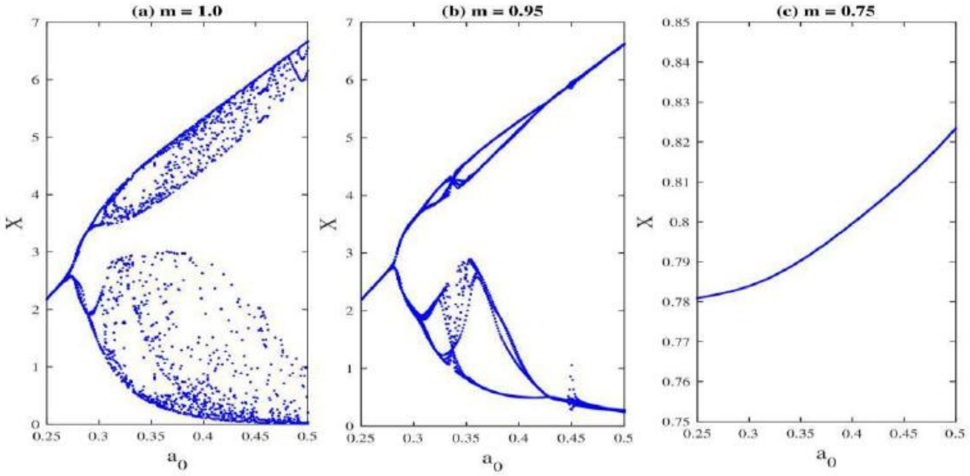

Figure 1: Bifurcation diagram of system (4) for the X population with respect to $a_0 \in (0.25, 0.5)$ with different fractional orders $m = 0.95, 0.75$ (Fig. 1(b) and 1(c)) and integer order $m = 1$ (Fig. 1(a)). Here $b_0 = 0.075, a_1 = 0.105, d_{1,2} = 10, d_3 = 20, v_0 = 1, v_1 = 2, v_2 = 0.405, v_3 = 1$ with $c_3 = 0.047$.

Example 2: I keep c_3 unaltered, and I choose a smaller value of $a_0 = 0.27$ (say) and remaining all parameters are taken from example 1. Initial values are indicated with stars and equilibrium values are denoted by red circles in the figure. Step size for all simulations is considered as 0.05. Using the above parameter set, I first verify the existence criteria of E^*. Here I observe $y^* - \frac{a}{s} = -1.4644 < 0$, $b < \frac{cx^*}{a + x^{*2}} = -0.7582 < 0$ and $v_3 - c_3 d_3 = 0.06 > 0$. Hence $E^* = (2.5772, 1.2766, 5.7002)$ exists in $\mathbb{R}^3_+$. Then compute $D(F) = -0.0084 < 0$, $A_1 = 0.4033 > 0$, $A_2 = 0.0689 > 0$, $A_3 = 0.0221 > 0$. Thus, following Theorem (3) (ii), the interior equilibrium E^* should stable for $0 < m < \frac{2}{3}$. In Fig. 2, I plot the time series solutions and phase portrait of FDE system (4) with different values of $0 < m = 0.65, 0.60 < \frac{2}{3}$. It shows that all populations remain stable for all values of $0 < m < \frac{2}{3}$, though solutions reach to equilibrium value more slowly as the value of m becomes smaller (see Figs. 2(a) - 2(d)).

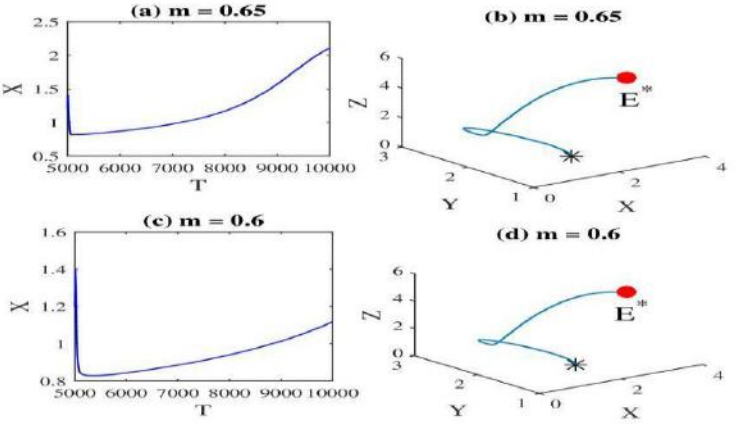

Figure 2: The trajectory and phase portrait of system (4) with different fractional orders $m = 0.95, 0.75$(Fig. 2(c) - 2(f)) and integer order $m = 1$ (Fig. 2(a) - 2(b)). I observe that unstable behavior of our system changes to stability with decreasing of fractional order m. All the parameters are same as in example 1 with a_0 = 0.47 and $c_3 = 0.047$.

Example 3: Again if I increase the value of a_0 = 0.35 and keeping all parameters unaltered as in example 1, I see that our system (4) exhibits 2-periodic limit cycle, 1-periodic limit cycle for higher values of fractional order $m = 0.85$ as well as for integer order $m = 1$ (see Figs. 3(a) - 3(d)). If I decrease the value of m, then limit cycle disappears and system becomes stable. Here I choose $m = 0.75$ and observe that solution converges to interior equilibrium point E^*= (4.0150, 1.2766, 0.7816, 5.6362) (see Figs. 3(e) - 3(f)).

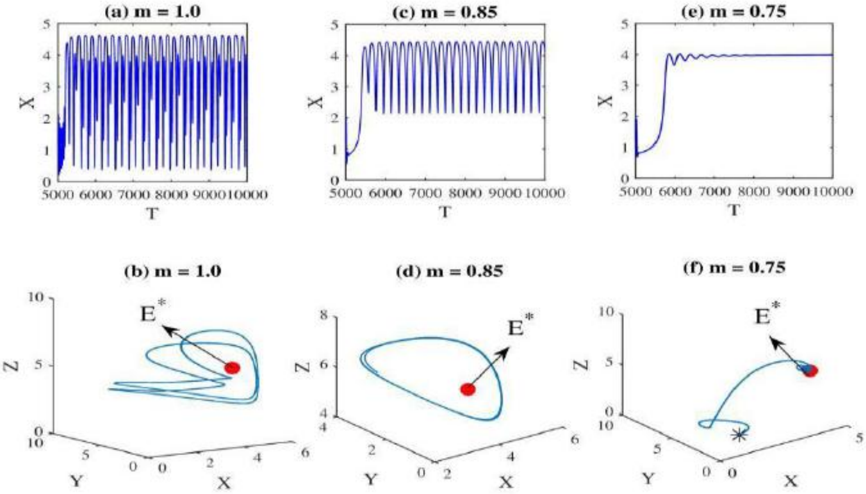

Figure 3: The trajectory and phase portrait of system (4) with different fractional orders $m = 0.85, 0.75$ (Figs. 3(c) - 3(f)) and for integer order $m = 1$ (Figs. 3(a) - 3(b)). I observe that the solution converges to interior equilibrium point for any values of $m < \frac{2}{3}$. It reaches to equilibrium value more slowly as the value of m becomes smaller. All the parameters are same as in example 1 with a_0 = 0.35 and $c_3 = 0.047$.

Example 4: Next I choose another parameter set $b_0 = 0.03, a_1 = 0.001, d_{1,2} = 10, d_3 = 20, v_0 = 0.85, v_1 = 2.5, v_2 = 2.2, v_3 = 1$ with $c_3 = 0.047$ and keeping same initial condition as in example 1, here I also choose a smaller value of a_0 = 0.15. Using the above parameter set, I first verify the existence criteria of E^*. Here I observe $y^* - \frac{a}{s}$ = -0.5532<0, $b < \frac{cx^*}{a + x^{*2}}$ = -2.6280 < 0 and $v_3 - c_3 d_3$ = 0.06 >0. Hence E^* = (3.2296, 1.2766, 2.0205)$ exists in $\mathbb{R}^3_+$. Then compute D(F) = -0.0217< 0, A_1 = -0.0131<0, $A_2 =$ -0.0044<0. Thus, following Theorem (3) (iii), the interior equilibrium E^* should unstable for $m > \frac{2}{3}$. In Fig. 4, I plot the time series for Y population and draw phase portrait of FDE system (4) on XY plane with different values of $m = 0.95, 0.85 > \frac{2}{3}$. It shows that Y population become unstable for different values of $m > \frac{2}{3}$ and our system (4) exhibits limit cycle around E^*. (see Figs. 4(a) - 4(d)).

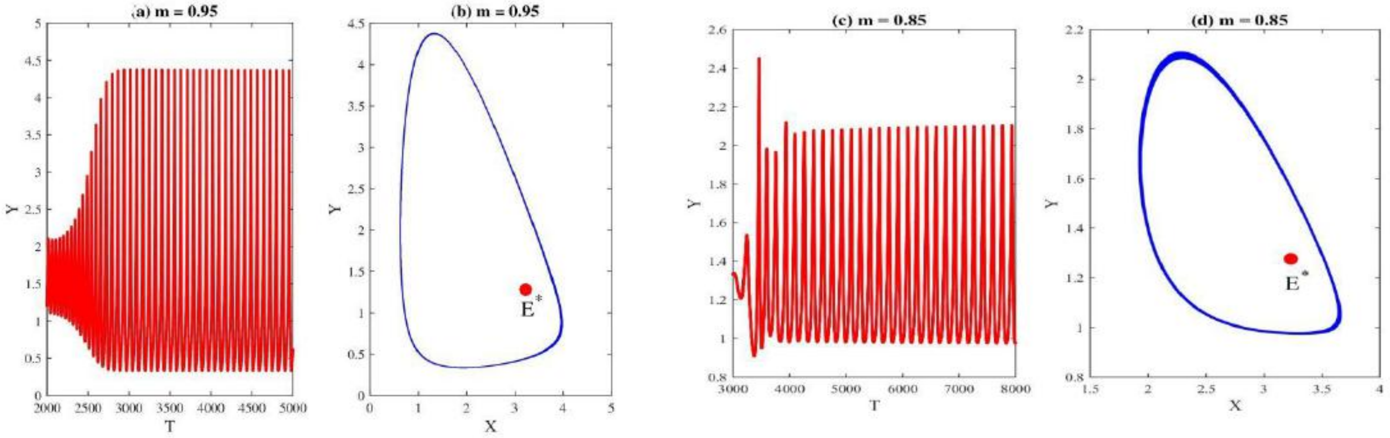

Figure 4: The trajectory and phase portrait of system (4) on XY plane with different fractional orders $m = 0.95, 0.85 > \frac{2}{3}$ (Figs. 4(a) - 4(d)). I observe that the Y population becomes unstable for different values of $m > \frac{2}{3}$ $b_0 = 0.03, a_1 = 0.001, d_{1,2} = 10, d_3 = 20, v_0 = 0.85, v_1 = 2.5, v_2 = 2.2, v_3 = 1$ with $c_3 = 0.047$ and initial values are same as in example 1 with $a_0 = 0.15$.

5. CONCLUSIONS:

In this paper, I extended the work of Mondal[34] on fractional order three-species food chain model with simplified Holling type IV functional response by giving the proof of local stability criterion of the interior equilibrium point. For local stability I used Routh-Hurwitz criterion for fractional order differential equations. Dissipativeness of the solutions is also proved in a feasible domain for fractional order system. It has been shown that all the solutions of system (4) are bounded above in a specific feasible domain. To confirm the analytical results of the system, numerical simulation is performed for different sets of biologically feasible parameter values. Simulation results also agree perfectly with the analytical results. Numerically, it has been observed that the fractional order system shows more complex dynamics, like chaos as fractional order becomes larger and shows more simpler dynamics as the order m decreases and becomes stable for lower value of m. Moreover, dynamics of the fractional-order system not only depends on system parameters but also depends on fractional order m.

Conflict of Interest: Author declares no conflict of interest.

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